

Filters In the Analog and Digital Domains

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FILTERS & LINEAR OSCILLATORS

Objectives

- **Understand the Laplace domain in a practical, intuitive sense.**
- **Understand the basics of analog filters.**
- **Learn about the continuum of passive filters, active filters, switched-capacitor filters and digital filters.**
- **Consider the trade-offs between filter types and implementations: size, weight, power, cost, flexibility.**



FILTER APPLICATIONS

- **Definition:** Filters amplify or attenuate components of signals on the basis of their frequency.

- **EXAMPLES:**

Audio equalizer

Variable gain or attenuation in several bands.

Audio bass or treble control

Variable gain or attenuation with two bands.

Oscilloscope trigger circuit

Attenuation of high or low frequencies (“noise”) to yield stable triggering.

Telephone voice-band filter

Limits signal frequencies to voice band ($\approx 300 - 3\text{kHz}$).

Interference filter

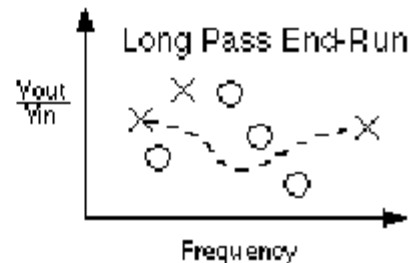
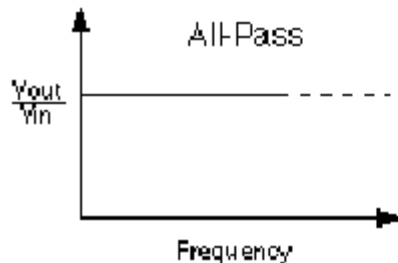
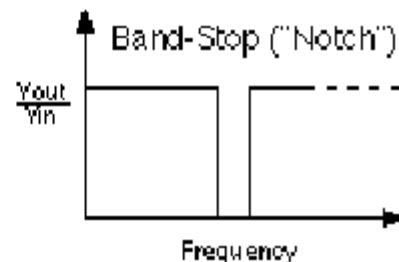
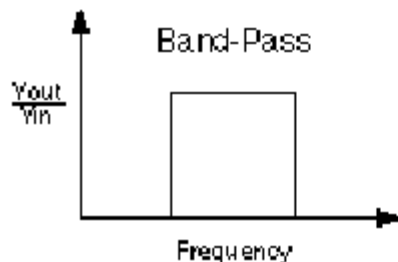
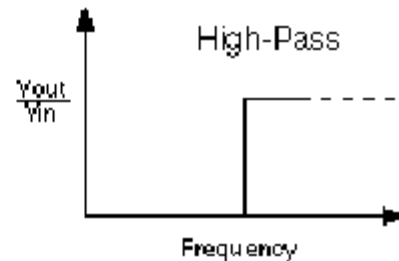
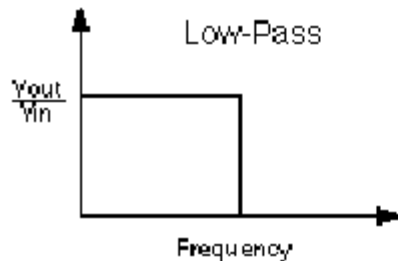
Limits interference from signals outside of frequency band of interest.

Anti-Aliasing filter

Rejects frequencies above the Nyquist rate for a mixed-signal system.


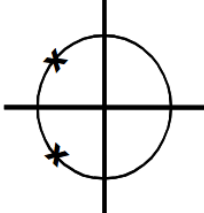

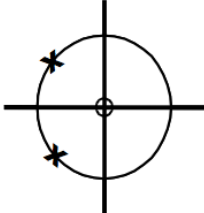
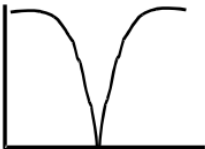
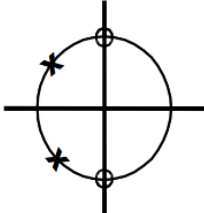

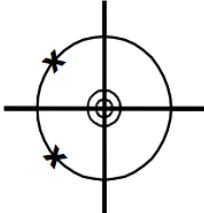
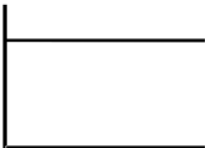
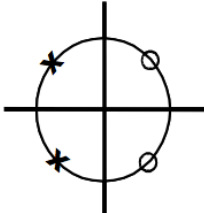


The Basic Types of Filter



- Real filters don't have "brick-wall" (infinitely steep roll-off) responses.
- You can implement them as passive (RLC) filters, active filters, switched-capacitor filters, or digital filters (DSP).
- We will consider and compare all four types and try to understand the trade-offs between them.



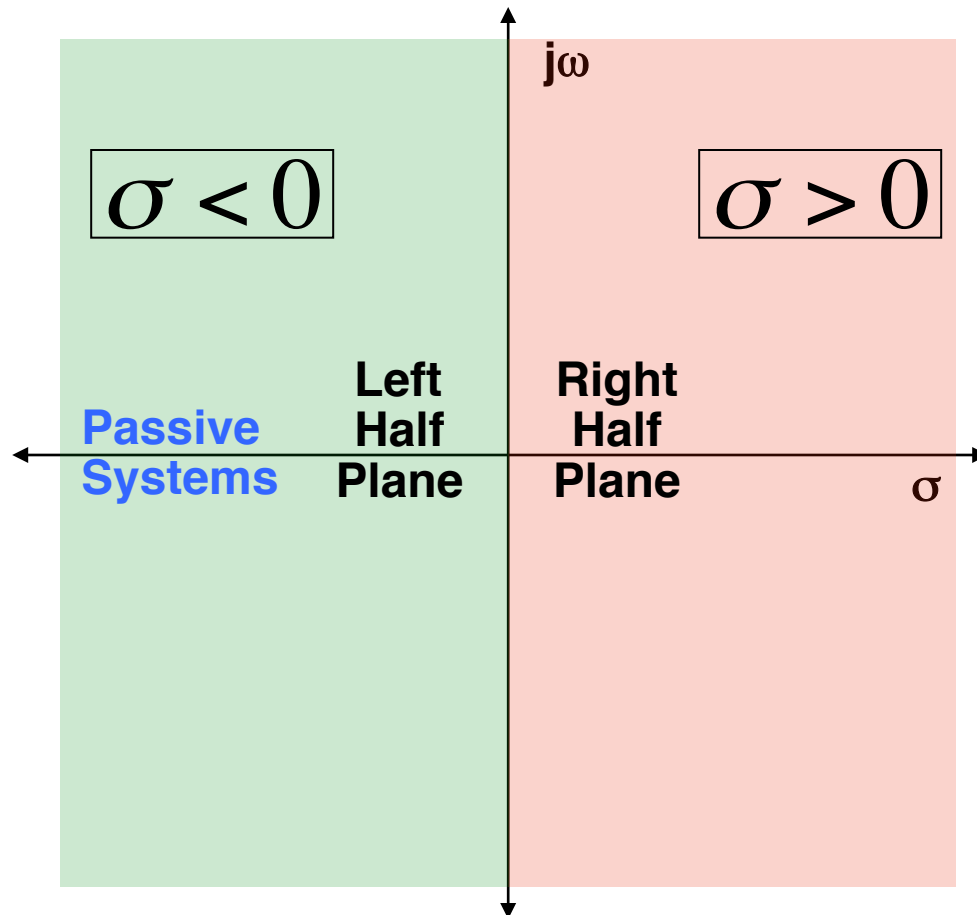
FILTER TYPE	MAGNITUDE	POLE LOCATION	TRANSFER EQUATION
LOWPASS			$\frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
BANDPASS			$\frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
NOTCH (BANDREJECT)			$\frac{s^2 + \omega_z^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
HIGHPASS			$\frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
ALLPASS			$\frac{s^2 - \frac{\omega_0}{Q}s + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$

Later, we will see how these filter shapes emerge from placement of poles and zeroes, and the underlying transfer functions in the Laplace domain.

<http://www.analog.com/library/analogdialogue/archives/43-09/edch%208%20filter.pdf>



Laplace Domain



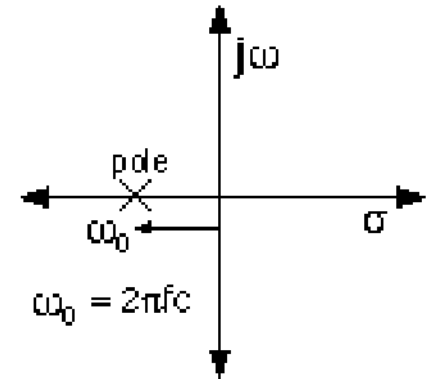
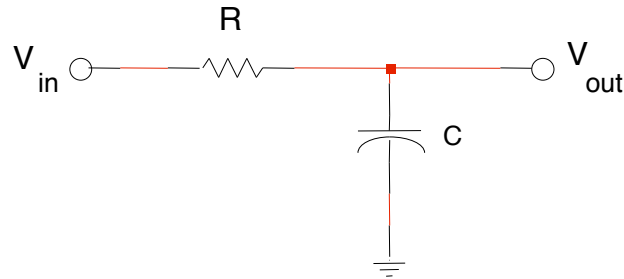
- Laplace allows us to represent more than Sinusoidal Steady State (output of a system in response to a “signal generator”).
- It allows us to represent signals that vary in amplitude over time, such as the decays of piano notes ($\sigma < 0$) and the out-of-control oscillations of bad control loops ($\sigma > 0$).
- We can also represent “perfect” steady state sinewave oscillators ($\sigma = 0$).

$$y(t) = e^{St} = e^{(j\omega + \sigma)t} = e^{j\omega t} e^{\sigma t}$$



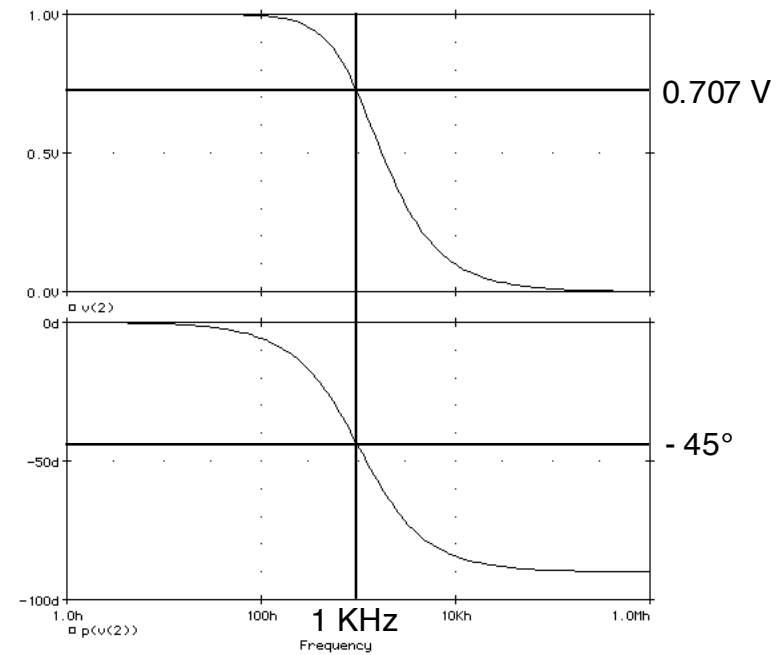
REVIEW: FIRST ORDER LOW-PASS

$$\frac{V_{out}}{V_{in}} = \frac{a_0}{s + \omega_0}$$



This is a pole at ω_0

$$\omega_0 = 2\pi f_c = \frac{1}{RC}$$

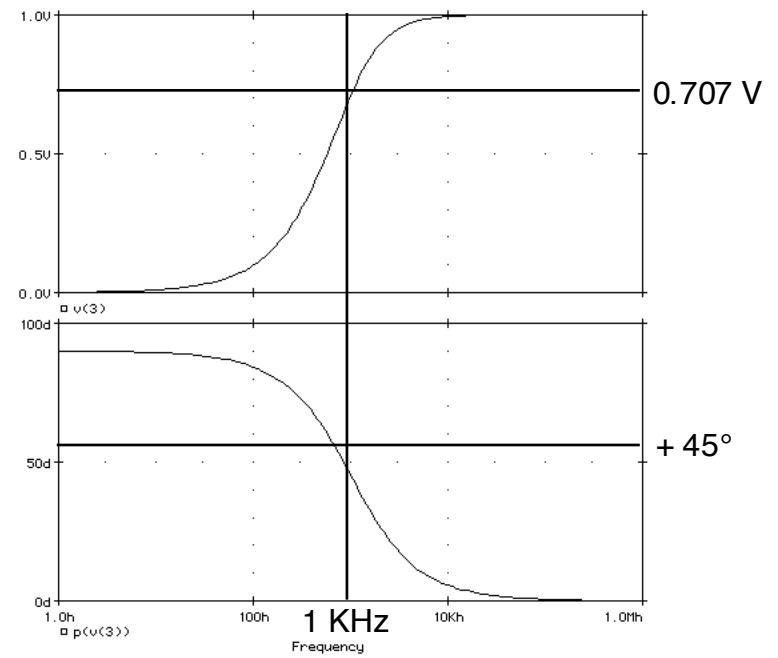
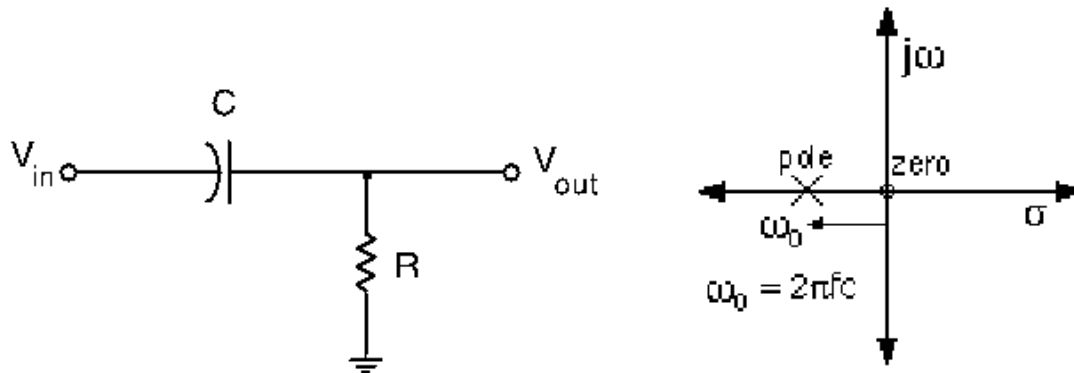


Review: First-Order High-Pass Filter

$$\frac{v_{out}}{v_{in}} = \frac{a_1 S}{S + \omega_0}$$

There is a pole at $S = -\omega_0$ and a zero at $S = 0$

$$\omega_0 = 2\pi f_c = \frac{1}{RC}$$



Poles, Zeroes, Filters...

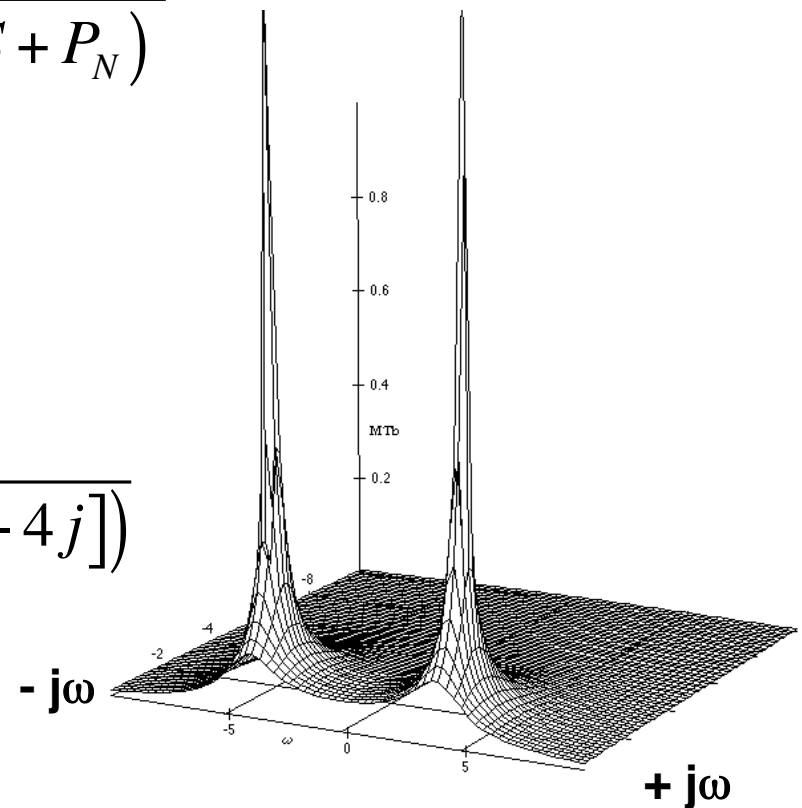
Can express general transfer functions as polynomials:

$$T(S) = \frac{v_{out}}{v_{in}} = \frac{Z(S)}{P(S)} = \frac{(S + Z_1)(S + Z_2) \cdots (S + Z_M)}{(S + P_1)(S + P_2) \cdots (S + P_N)}$$

Example: second-order low-pass filter.

$$T(S) = \frac{v_{out}}{v_{in}} = \frac{Z(S)}{P(S)} = \frac{A}{(S + [1 + 4j])(S + [1 - 4j])}$$

Note the complex conjugate pole pair.



Second-Order High-Pass Filter

Example: second-order high-pass filter.

$$HPF(S) = \frac{v_{out}}{v_{in}} = \frac{S^2}{(S + [1 + 4j])(S + [1 - 4j])}$$

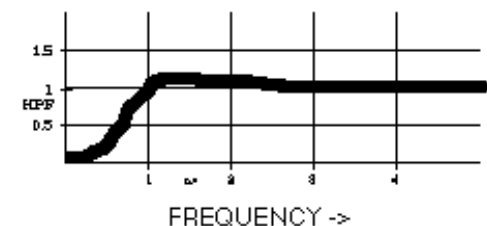
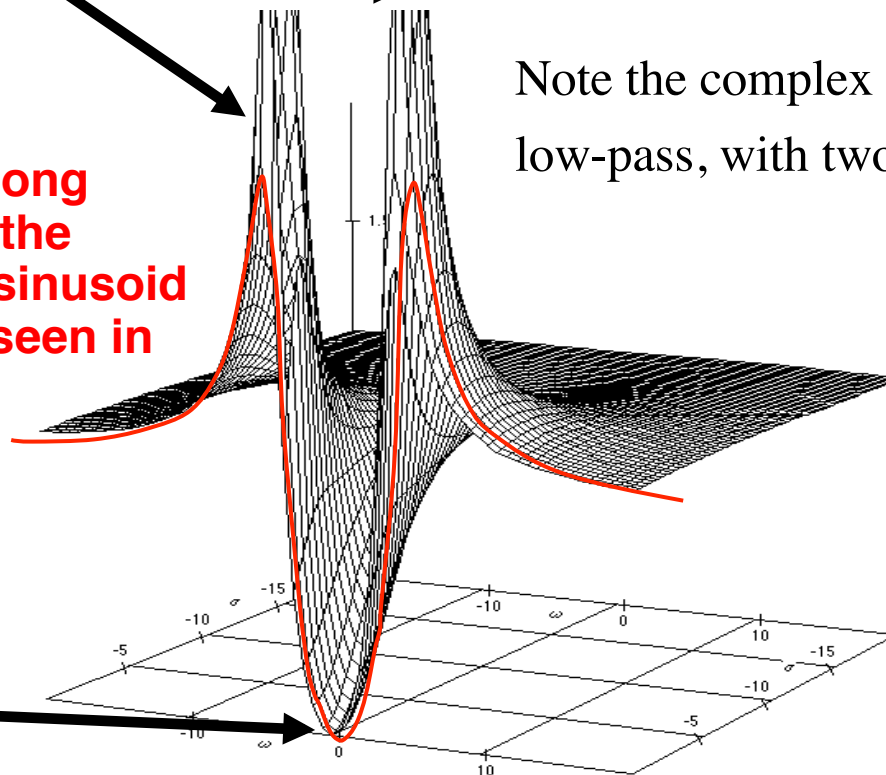
Two poles

KEY POINT:

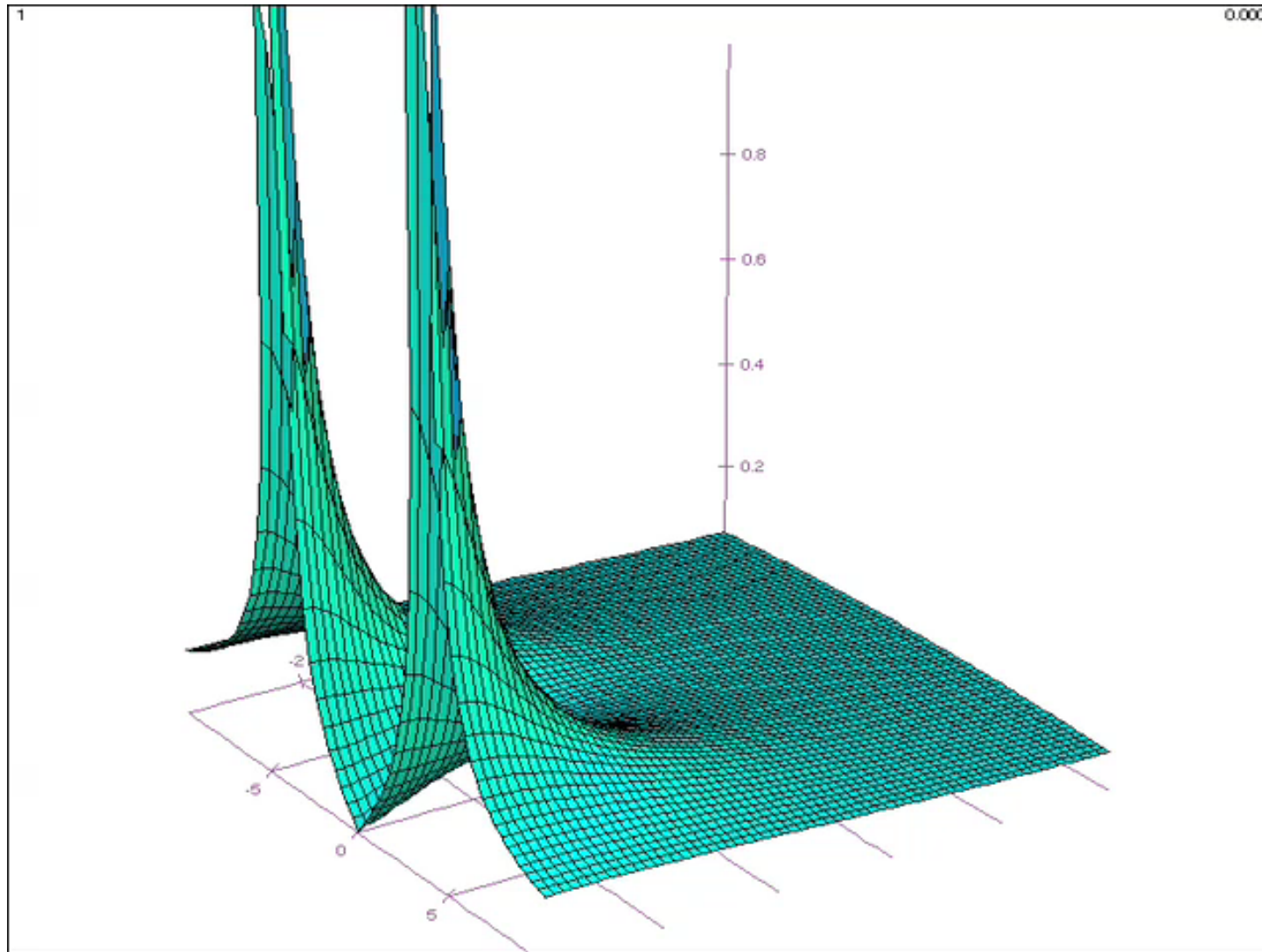
The “slice” along the $j\omega$ axis is the steady-state-sinusoid response as seen in Bode plots.

Note the complex conjugate pole pair, same as the low-pass, with two zeroes added.

Two zeros

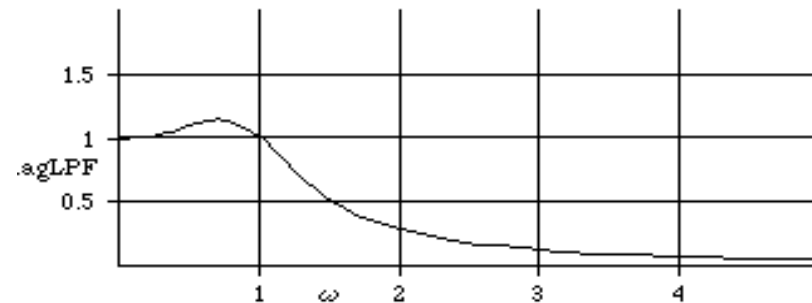


Second-Order High-Pass Filter

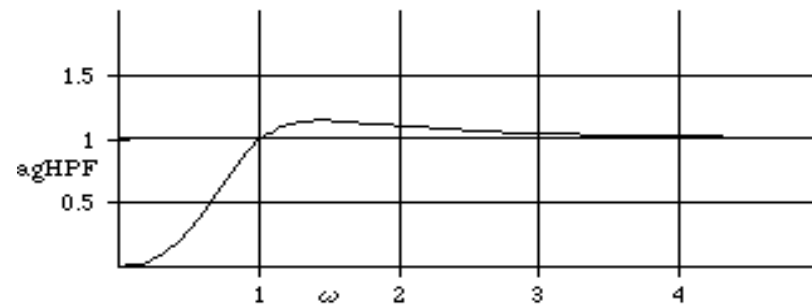


Basic 2nd Order Responses

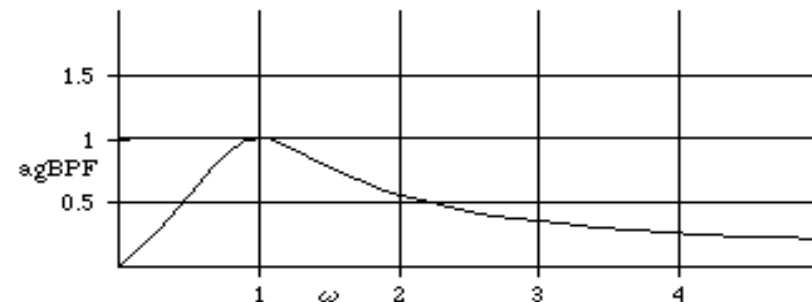
$$LPF(S) = \frac{A}{S^2 + \frac{\omega_0}{Q}S + \omega_o^2}$$



$$HPF(S) = \frac{AS^2}{S^2 + \frac{\omega_0}{Q}S + \omega_o^2}$$



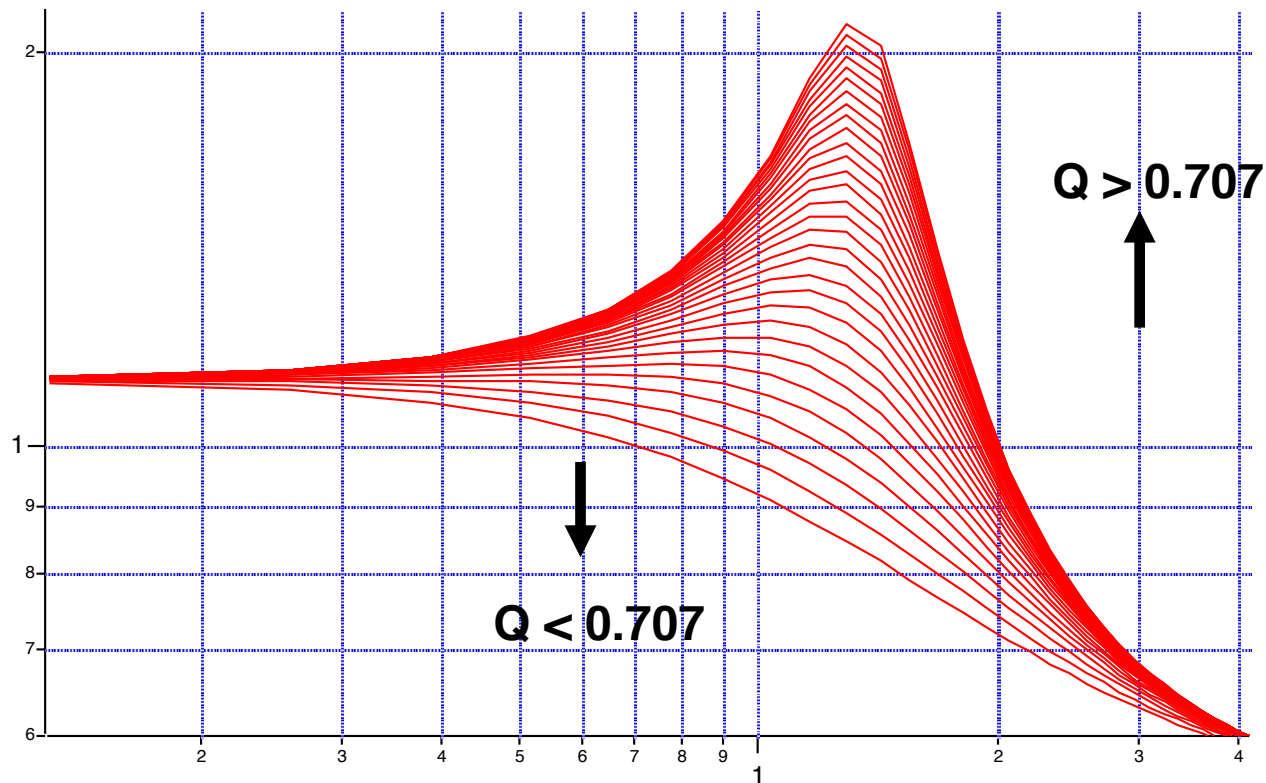
$$BPF(S) = \frac{AS}{S^2 + \frac{\omega_0}{Q}S + \omega_o^2}$$



Note: roll-off only -20dB/dec for BPF.



Quality Factor: Q

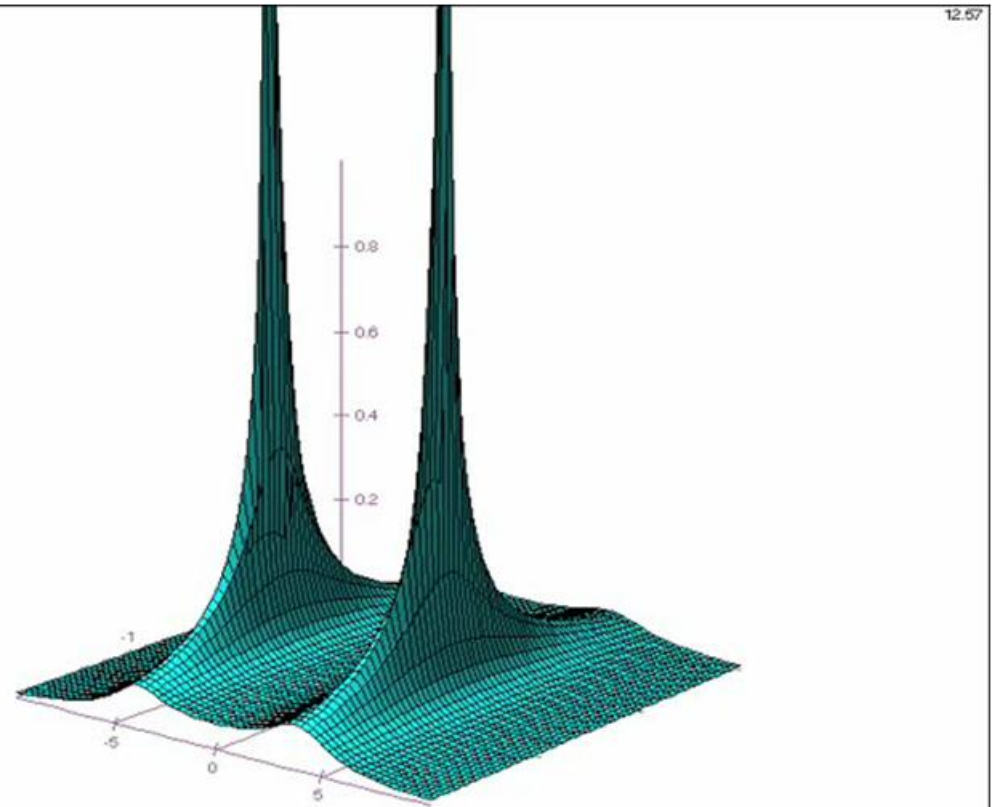
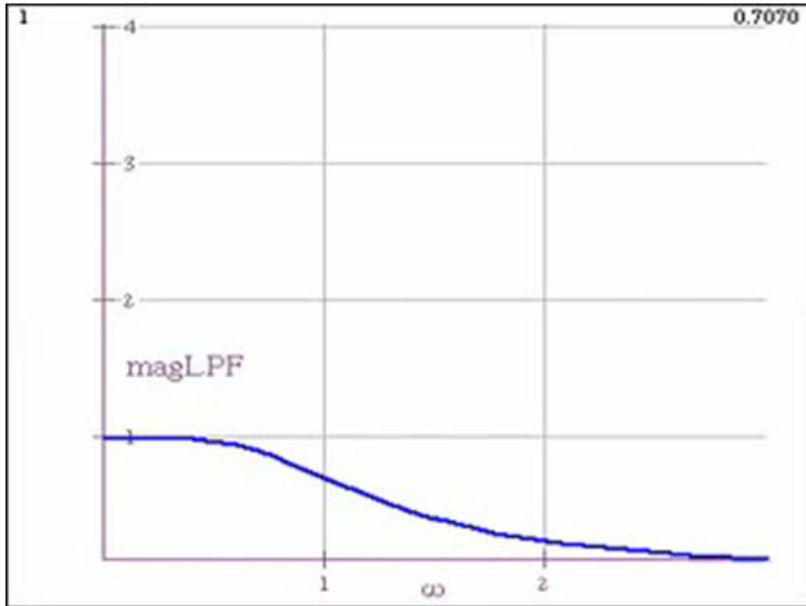


Q is a measure of how “resonant” or “peaked” a filter’s response is, derived from the original RLC filters that resonated like a tuning fork.

High Q means poles close to the $j\omega$ axis.

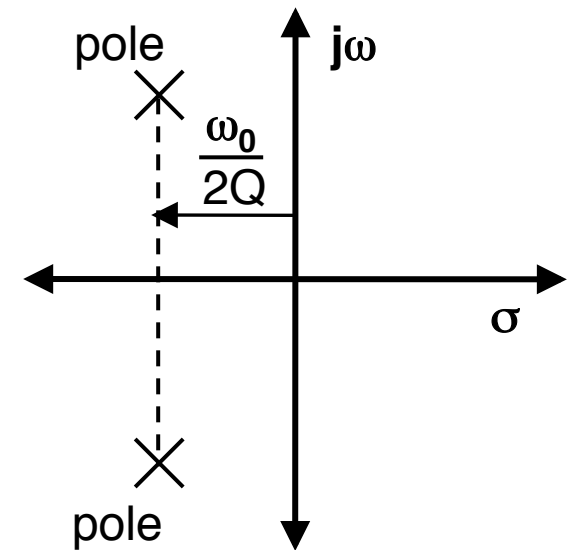


Second-Order Low-Pass Filter



Poles of Second-Order Systems

$$\text{LPF}(S) = \frac{A}{S^2 + S\frac{\omega_o}{Q} + \omega_o^2}$$



$$\text{POLE DISTANCE FROM } j\omega \text{ AXIS} = \frac{\omega_o}{2Q}$$

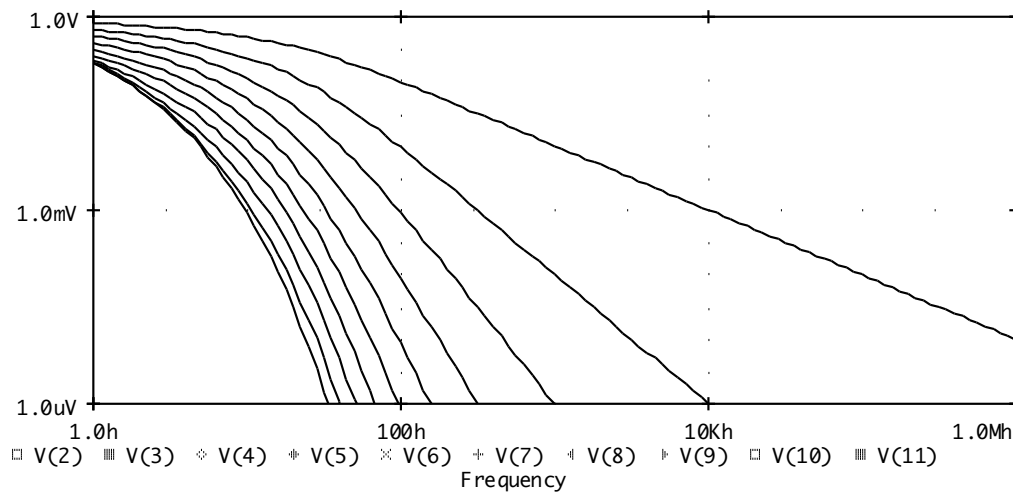
**FOR LEAST PEAKING
("MAXIMALLY-FLAT")**

$$Q = \frac{1}{\sqrt{2}}$$

$$P_{1,2} = -\frac{\omega_o}{2Q} \pm j\omega_o \sqrt{1 - \frac{1}{(2Q)^2}}$$



Cascading Filters for Better Performance

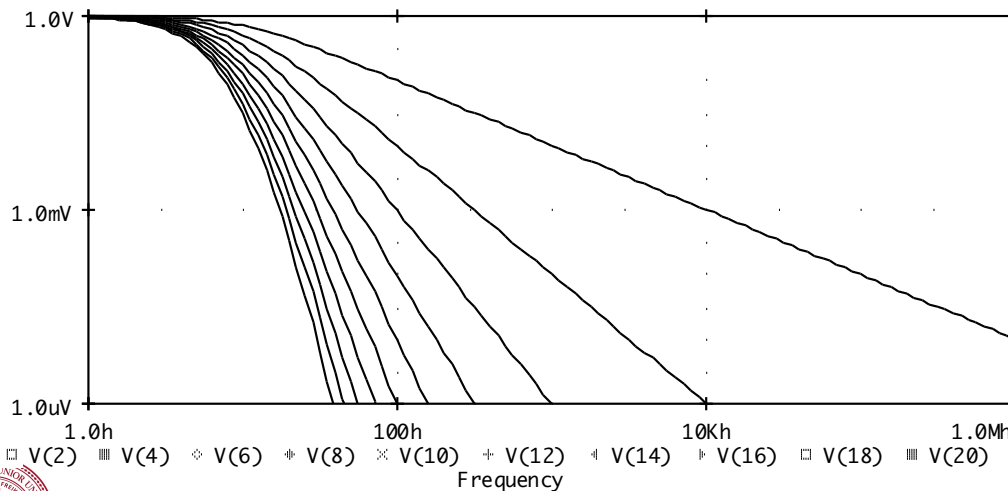


Directly cascading (putting in series) RC filters can lead to a lot of attenuation.

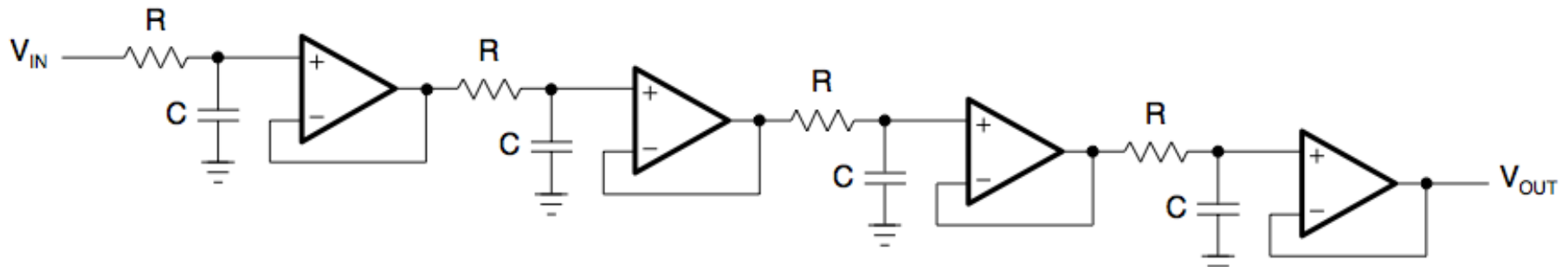
LC filters can be designed to minimize this.

Buffering between stages keeps the poles from interacting – use amplifiers such as op-amps.

One can do better if the poles are not all in the same place on the S-plane, but need to know how to arrange them.



Buffered Cascaded RC LPF



$$H(S) = \frac{1}{(1 + \alpha_1 S)(1 + \alpha_2 S) \cdots (1 + \alpha_N S)}$$

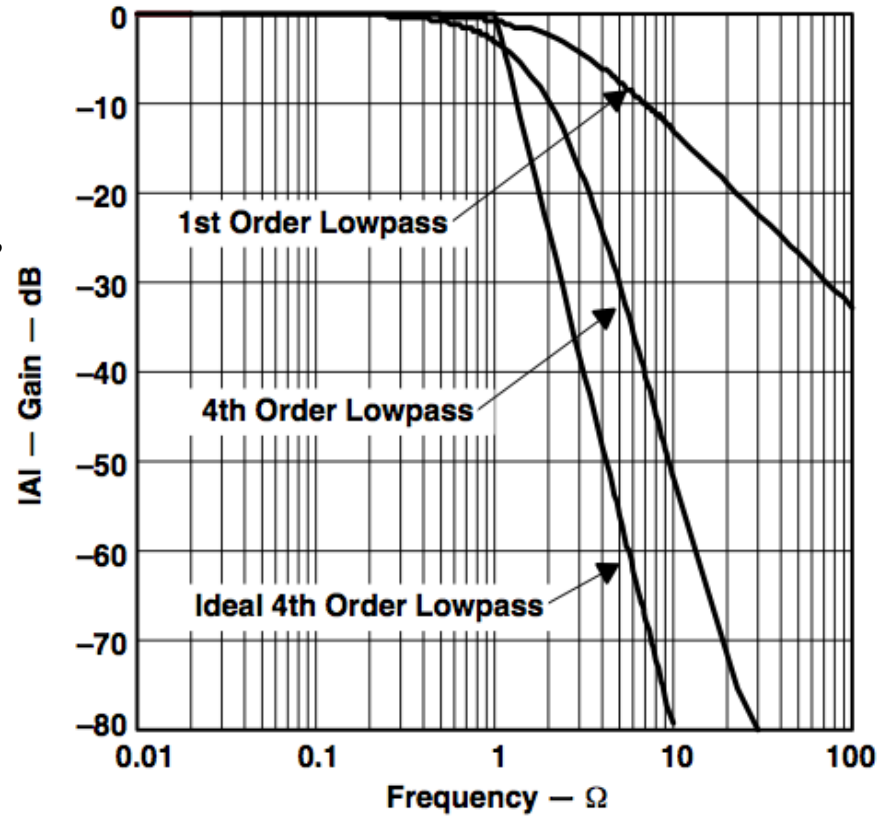
All sections have same cutoff frequency, f_c , then,

All coefficients, α_i become,

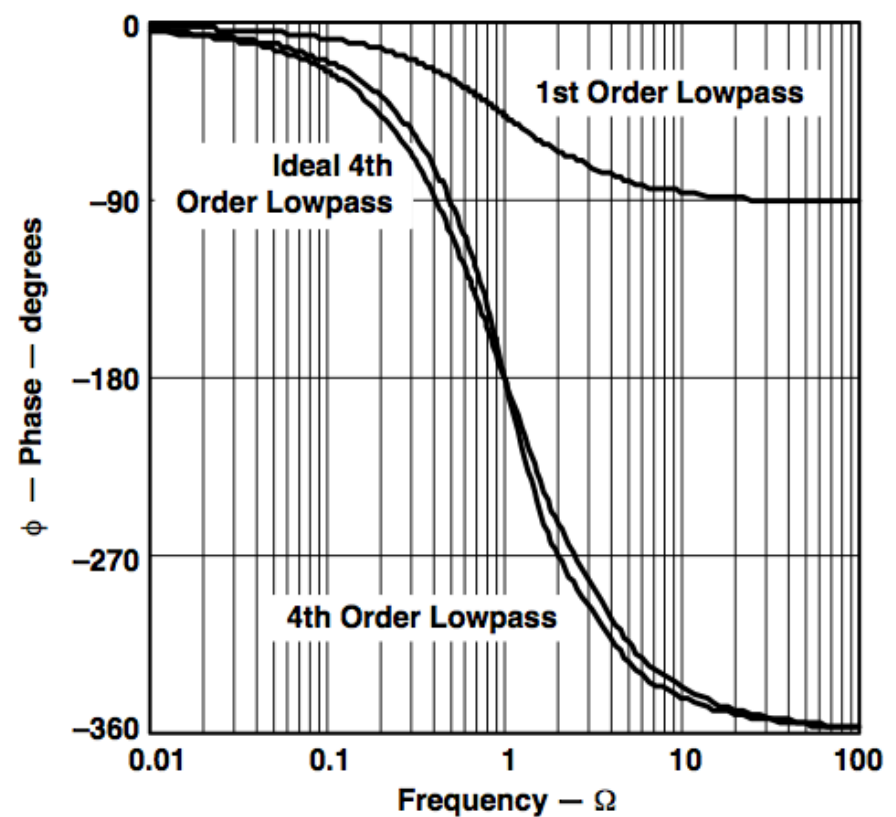
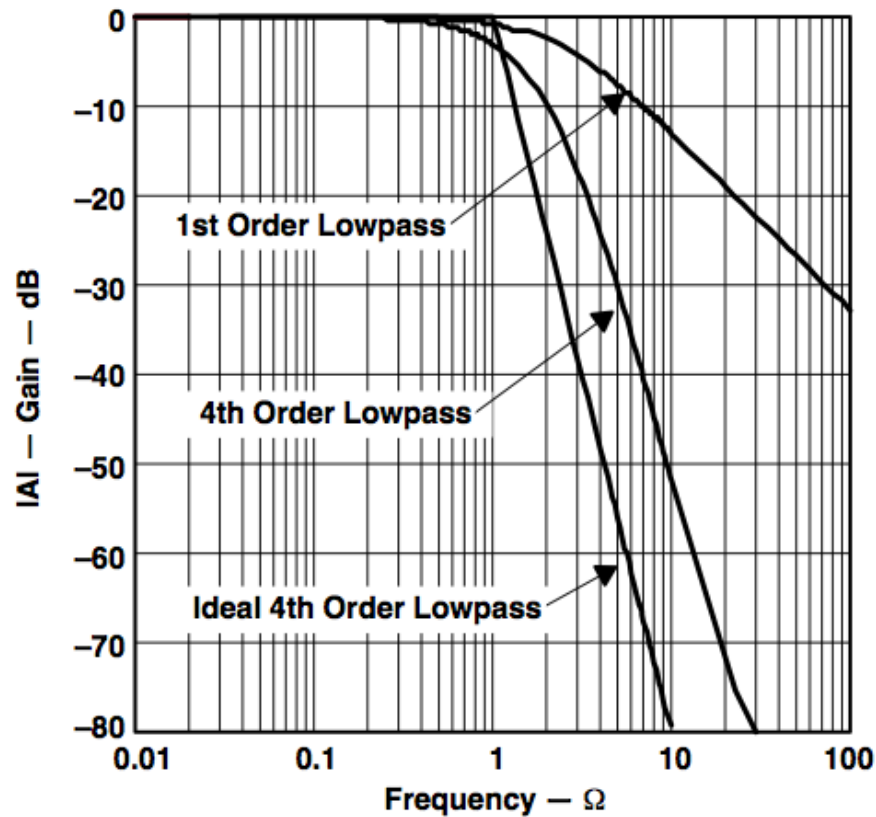
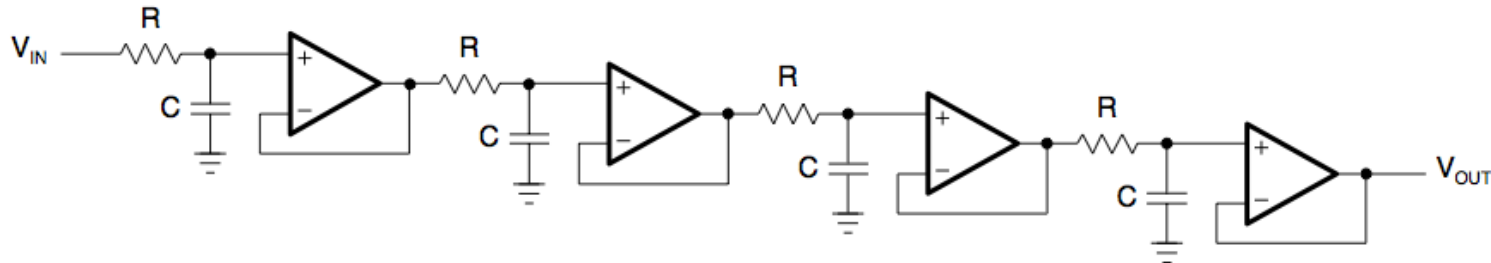
$$\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha = \sqrt{N/2} - 1$$

and f_{ci} of each sub-filter is $\frac{1}{\alpha}$ times the overall f_c

<http://www.ti.com/lit/ml/sloa088/sloa088.pdf>



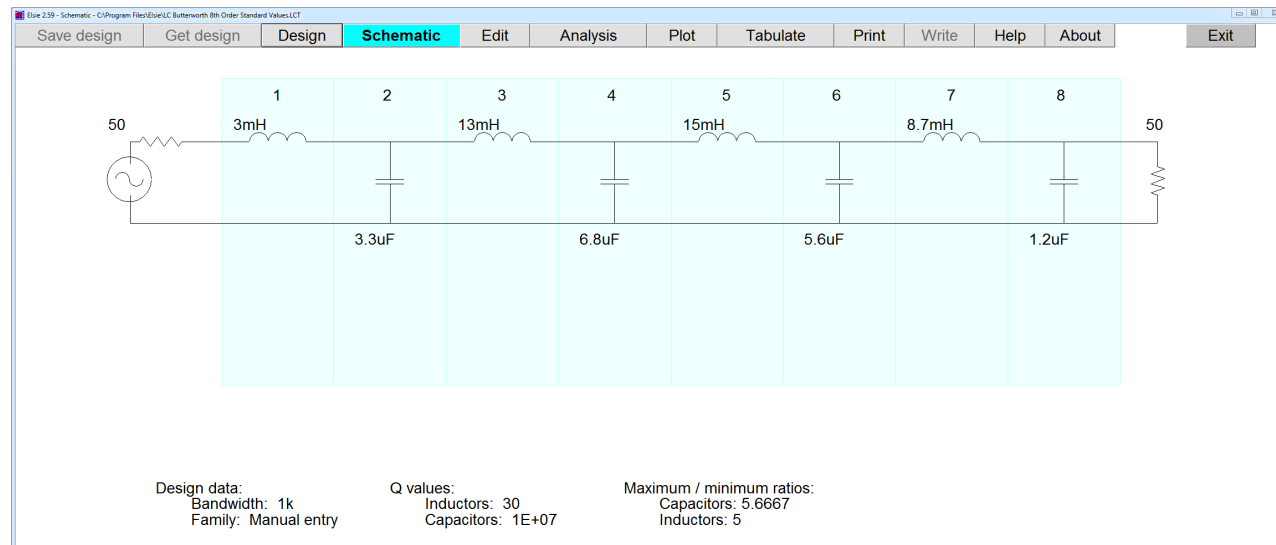
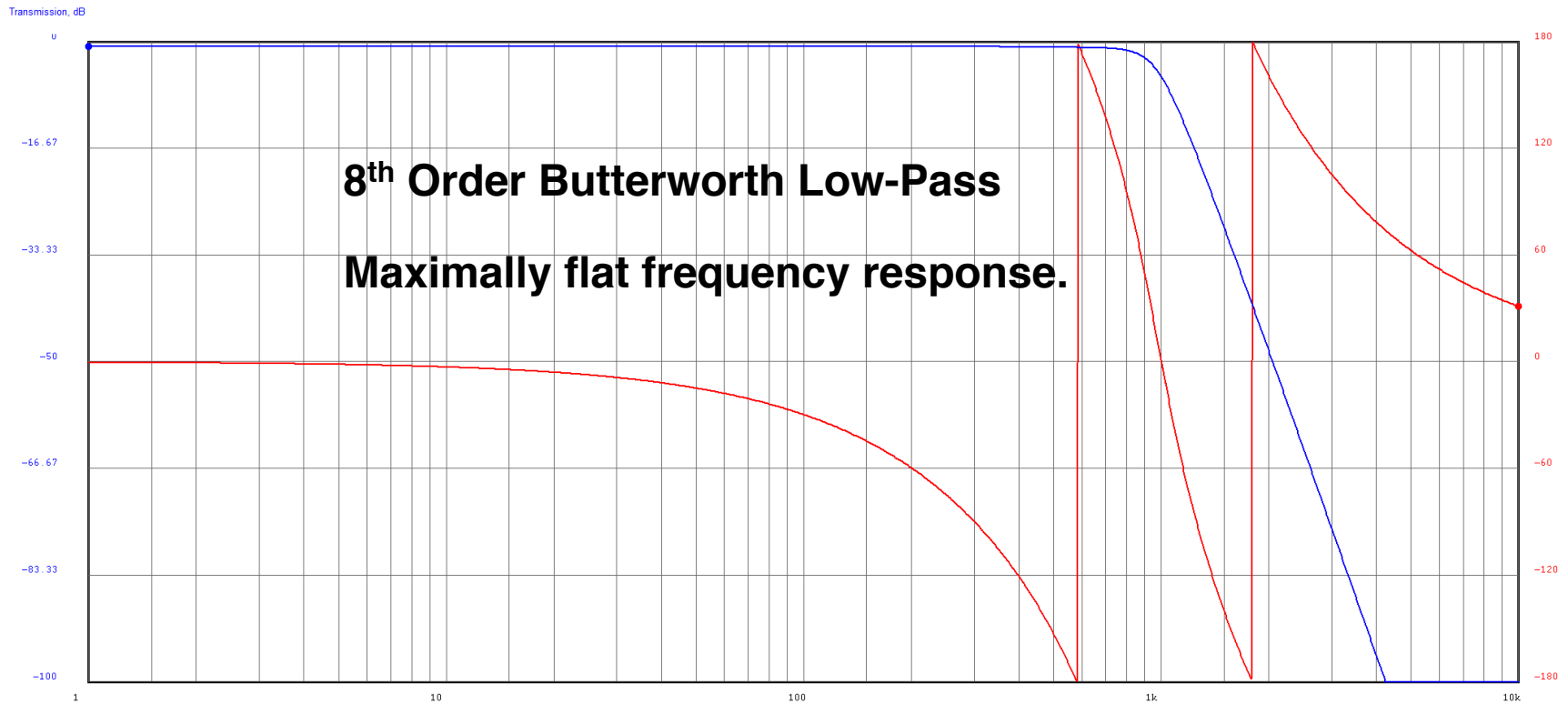
Buffered Cascaded RC LPF

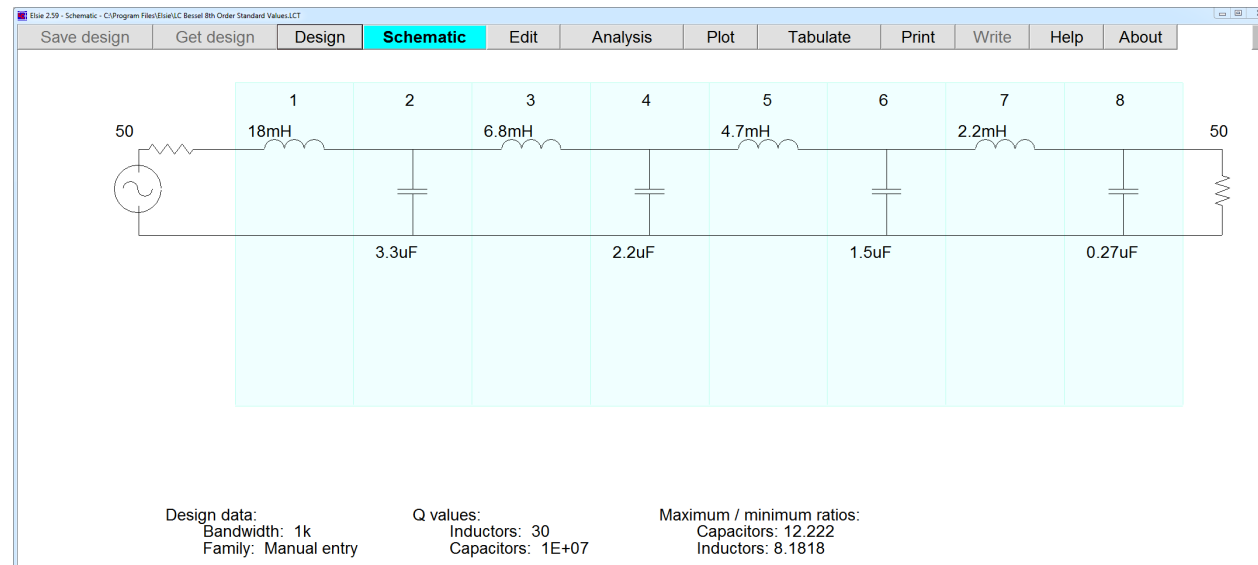
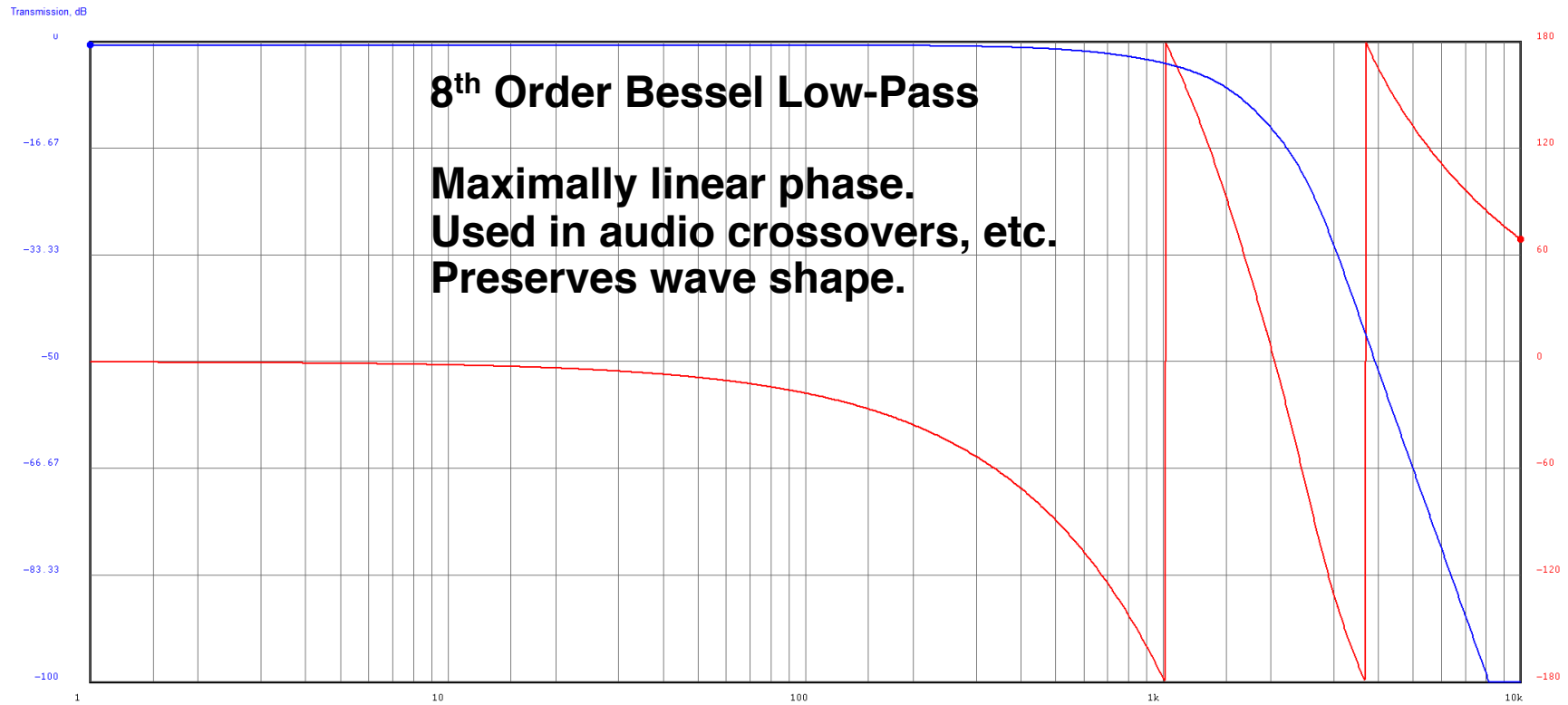


LC Filters

- With simple, passive components, very low-noise, high-dynamic range filters can be implemented.
- While they do not require power, they can be expensive, heavy and hard to tune if necessary.
- However, they are valuable in many applications.
- A very useful tool for designing them can be found here:
<http://tonnesoftware.com/elsie.html>
- The student version is free, and very powerful, handling up to 7th order filters.

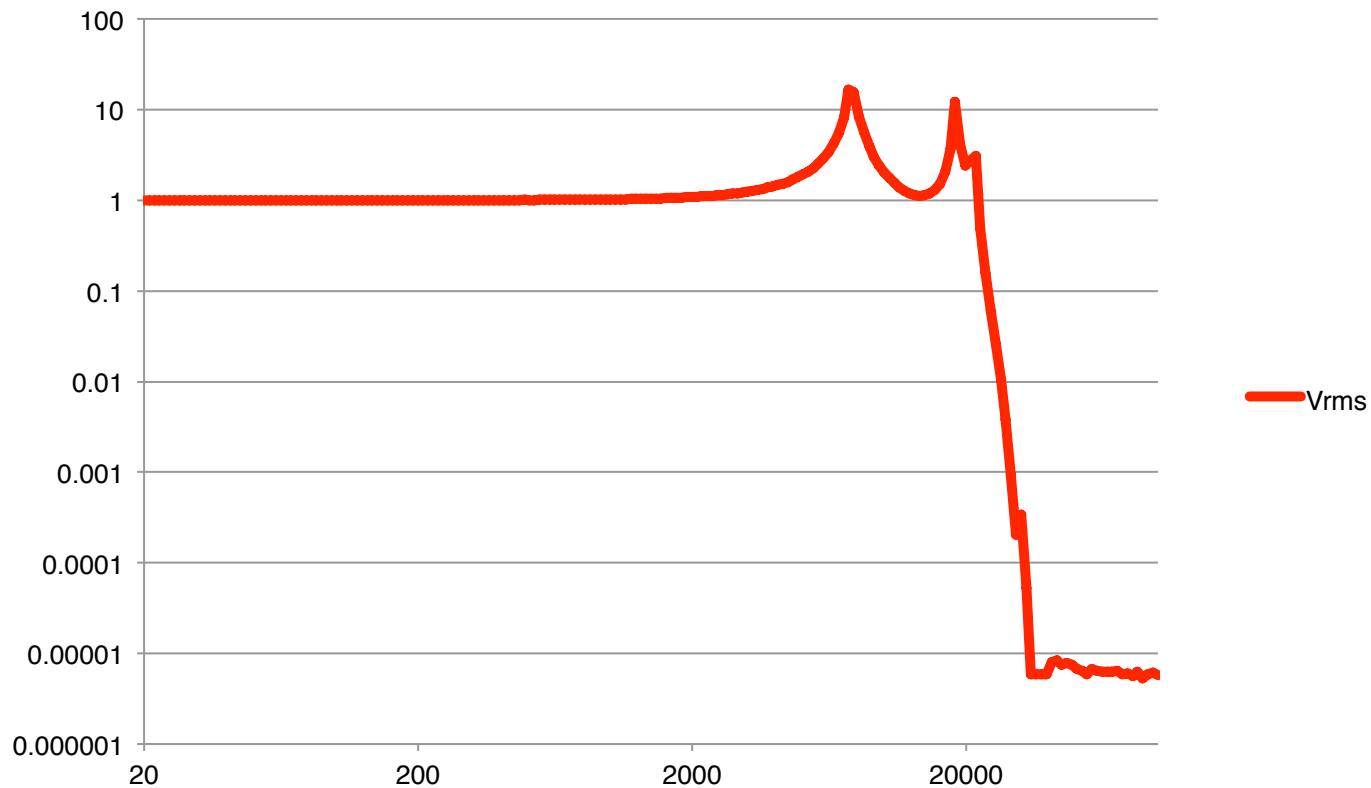






Example Complex LC Filter

7th Order Passive Low-Pass Filter Response (SR1)



Elliptical (Model TTE J77E), 7-Poles, 20 kHz f_c .

<http://www.tte.com/pdfs/data%20-%20tte%20designed%20lowpass%20filters%20-%20100hz%20to%20200mhz.pdf>



Introduction to Active Filters

Pro' s of Active Filters

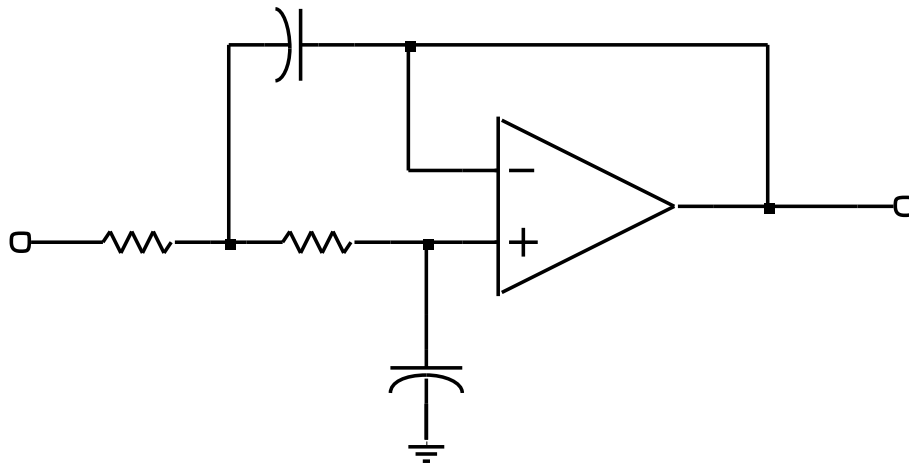
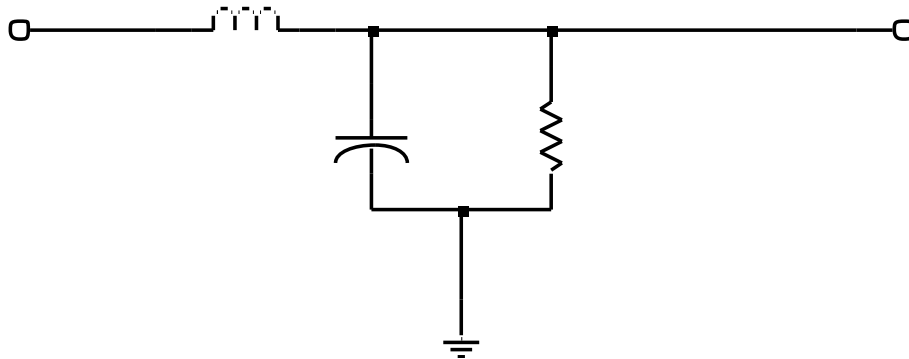
- 1) elimination of inductors
- 2) low cost (largely due to item 1)
- 3) smaller size and weight (due to item 1)
- 4) high isolation (high input impedance, low output impedance)
- 5) characteristics relatively independent of loading (due to item 4)
- 6) user-defined gain

Con' s of Active Filters

- 1) requirement for power
- 2) limited dynamic range (lower limit due to noise, upper limit due to clipping)
- 3) limited frequency range (lower limit due to large capacitors, higher limit due to active device performance)
- 4) noise greater than that of passive filters



Replacing the Inductors with Op-Amps

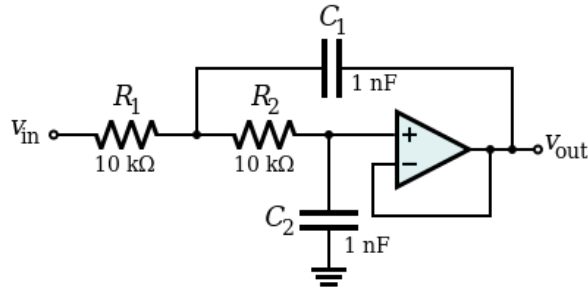


- Active filters can be designed using the principle that each inductor is replaced with an op-amp equivalent circuit.
- Feedback allows inductors to be simulated using capacitors.
- Inductors are heavy and expensive, so this approach makes sense!
- The type of active filter we will consider here is not based directly on inductor replacement, but can get an equivalent response.

Sallen & Key Topology

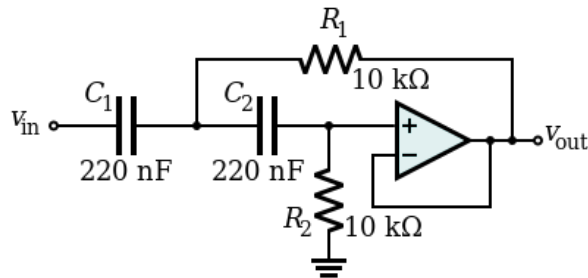


Sallen & Key and VCVS Blocks



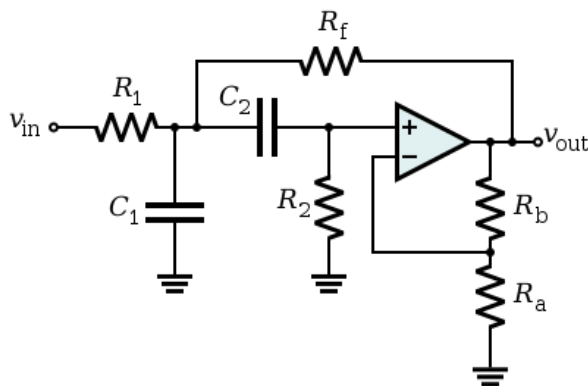
LPF

A low-pass filter, which is implemented with a Sallen–Key topology, with $f_c = 15.9$ kHz and $Q = 0.5$.



HPF

A specific Sallen–Key high-pass filter with $f_c = 72$ Hz and $Q = 0.5$.



BPF

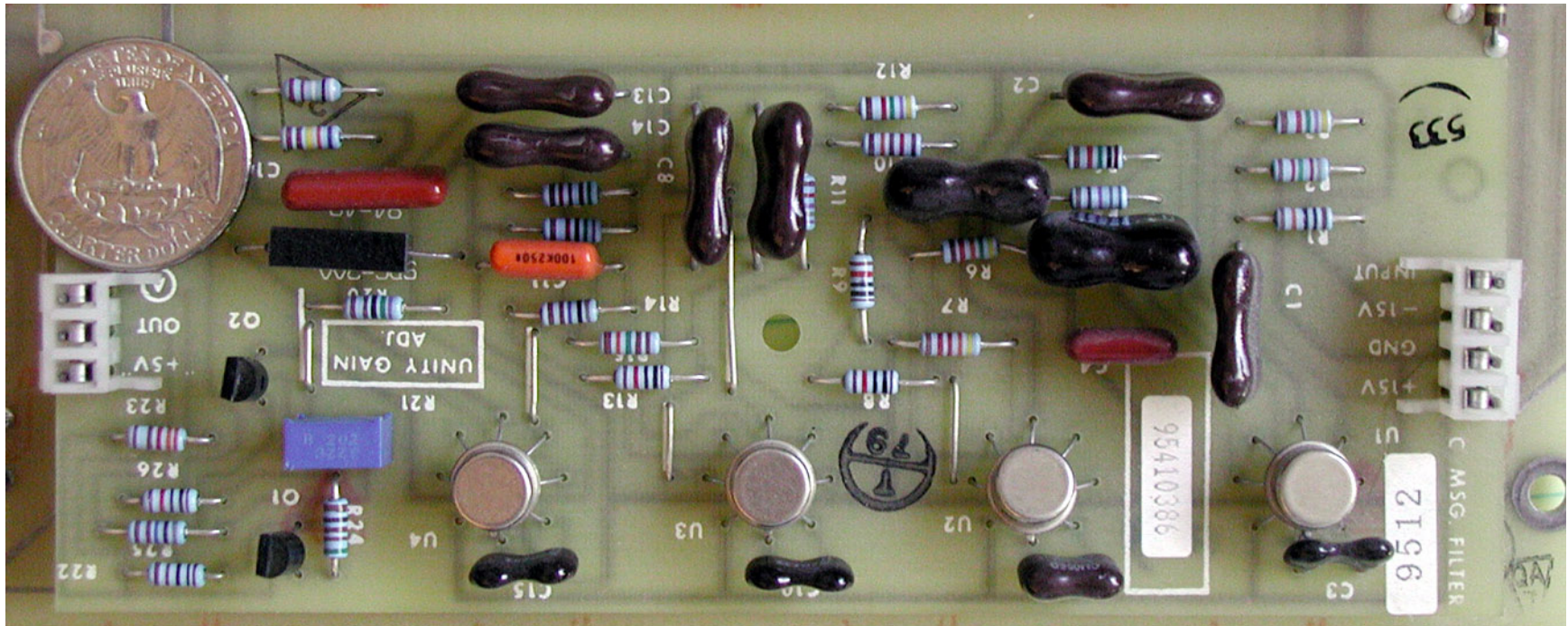
A bandpass filter realized with a VCVS topology.

http://en.wikipedia.org/wiki/Sallen–Key_topology



Multi-Stage Filters

- Using a variety of software tools (a free one is described later), you can arrange the poles of filters to give much sharper cutoffs, etc.
- The basic idea is to put a bunch of second-order filters in “series” - in a linear system, the order shouldn't matter.



Example multi-stage filter from the late 1970' s - still relevant, and the discretes are easy to see (not surface mount).



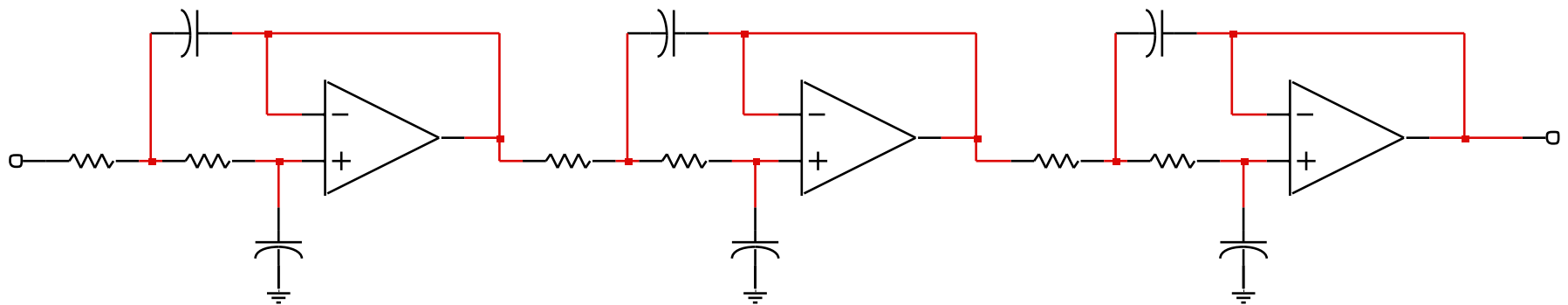
More Complex Active Filters

- Take advantage of good isolation between stages
- Build up the filter from a series of second- (and sometimes first-) order sections

The stages do not interact.

- Position the poles on the s-plane to get the desired response on the $j\omega$ axis...

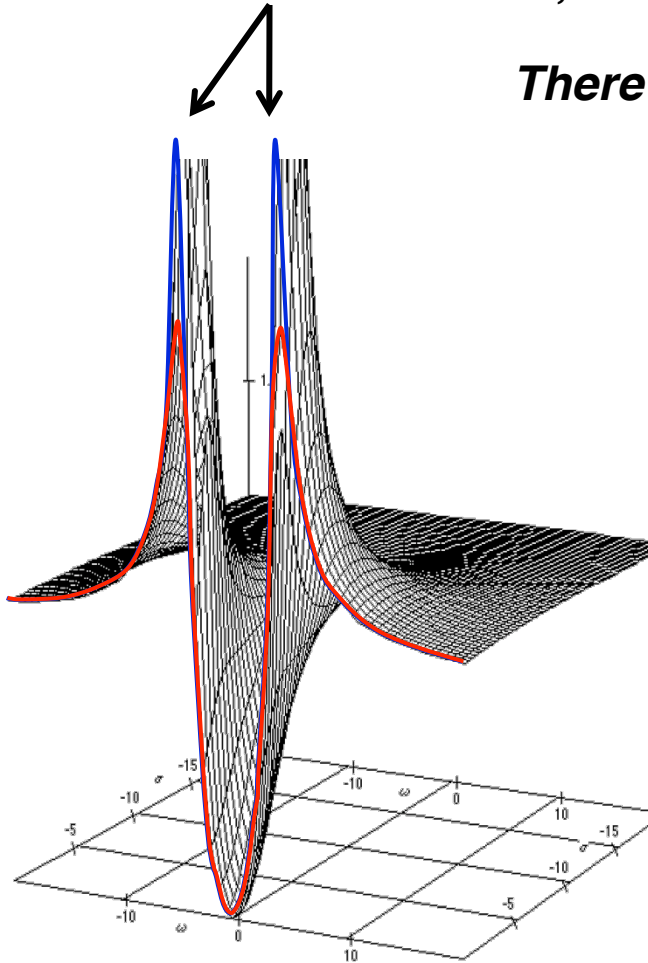
Each stage below is of the common Sallen & Key design (low-pass form shown).



Multistage Filters - Pole Position

Stacking more poles at the same place will increase roll-off, but also Q , which may provide an overly peaked response.

There are better ways to place poles.



A lot of work has been done to mathematically design multi-pole filters to achieve optimized (in one way or another) responses in frequency or time domains.

Of course, there are trade-offs when optimizing one parameter or another, and certain filter types (topologies) are better for certain applications or others.

At the level of EE122A, a high-level understanding is offered but students are obviously free to delve into the deeper theory and math on their own.

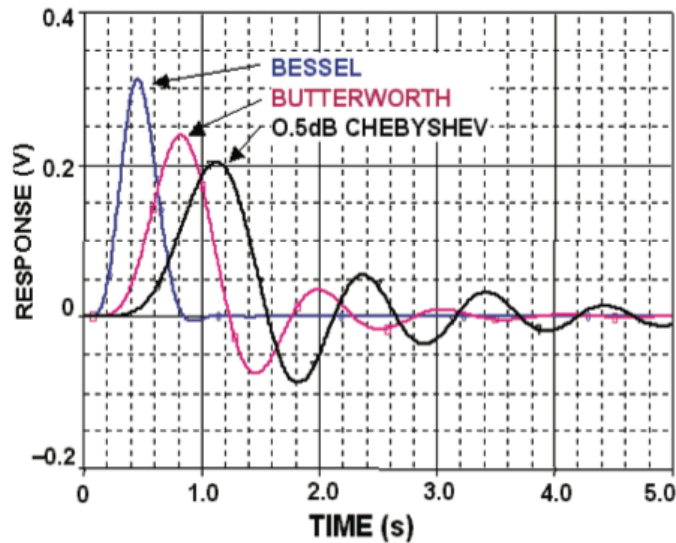
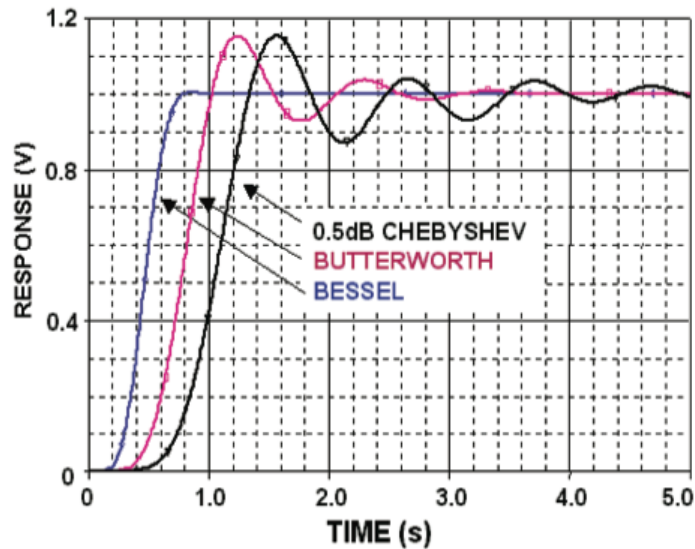
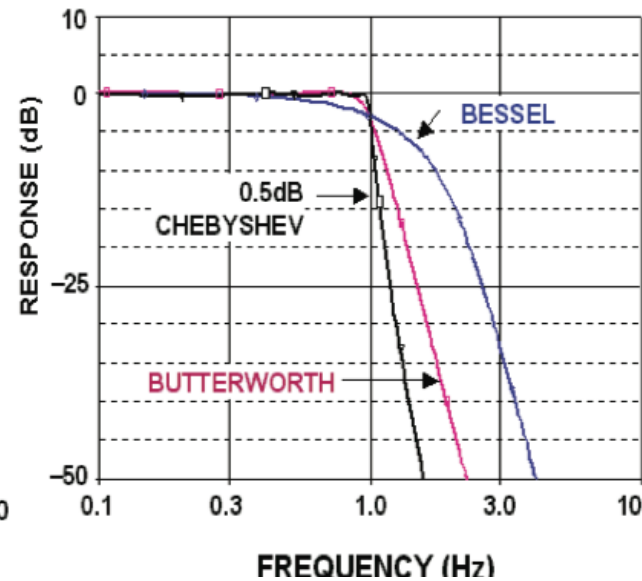
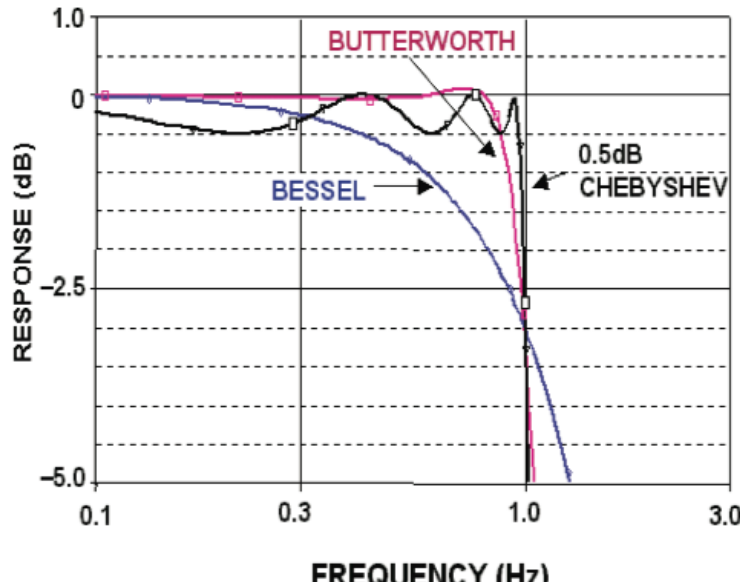


Example Low-Pass Filter Topologies

- Three (of many) example filter topologies and their optimizations are:
- **Butterworth**
 - Flattest pass-band.
- **Chebyshev**
 - Fastest roll-off from passband to stopband
- **Bessel**
 - Most linear phase response up to f_c



Example Low-Pass Filter Topologies



Butterworth Filters

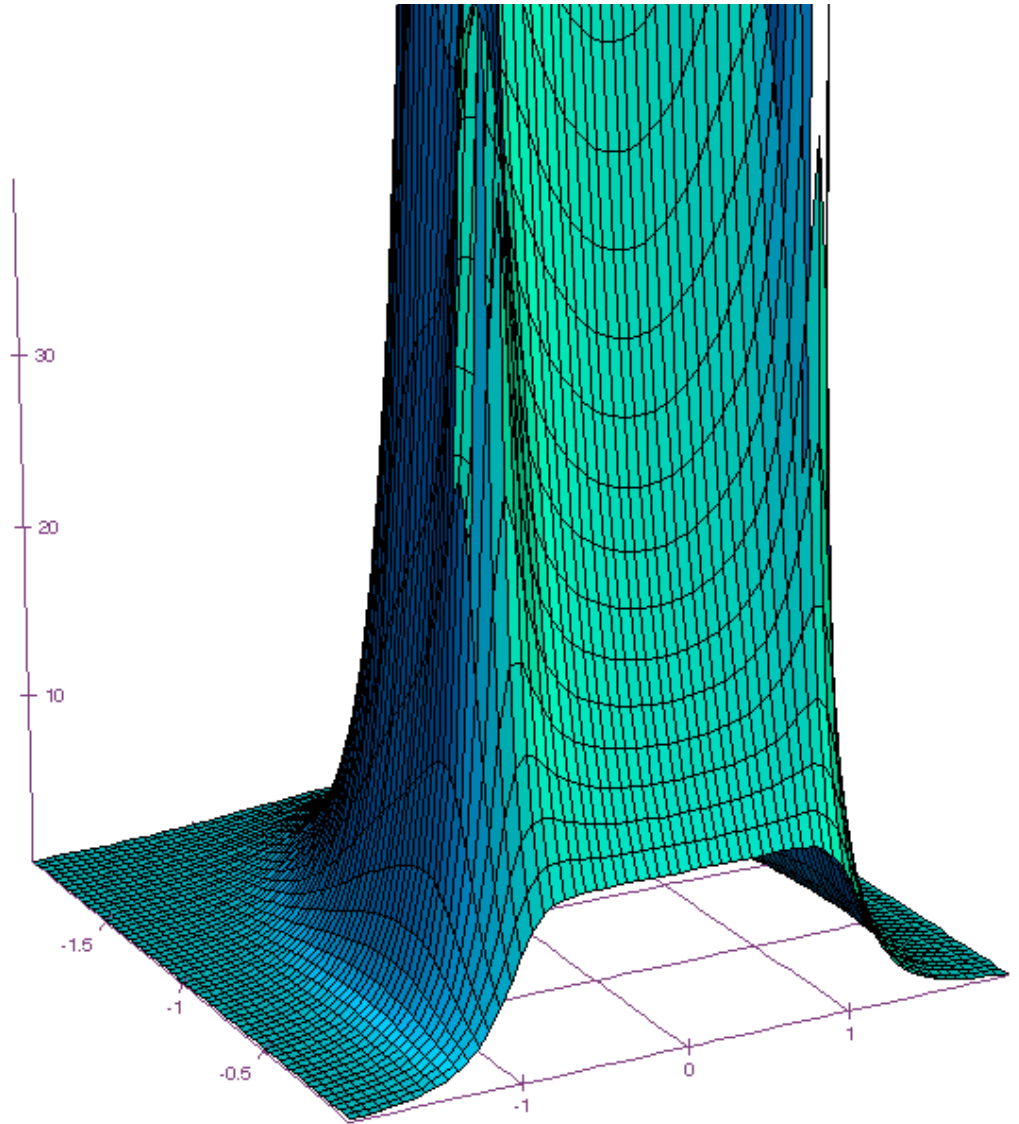
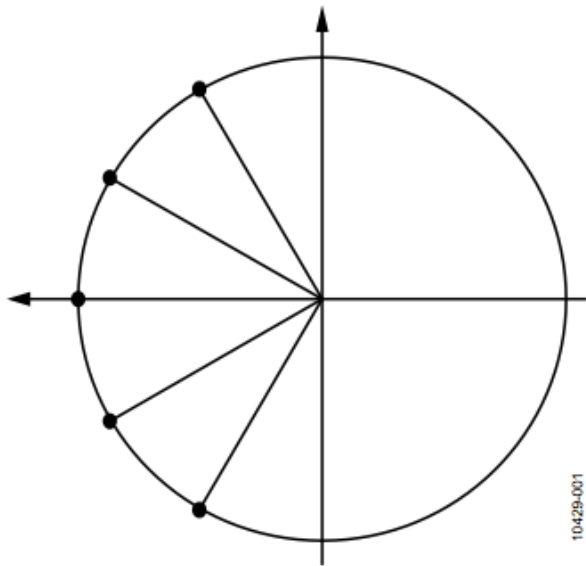
Poles arranged on a unit circle at:

$$-\sin \frac{(2K-1)\pi}{2N} \pm j \cos \frac{(2K-1)\pi}{2N}$$

where,

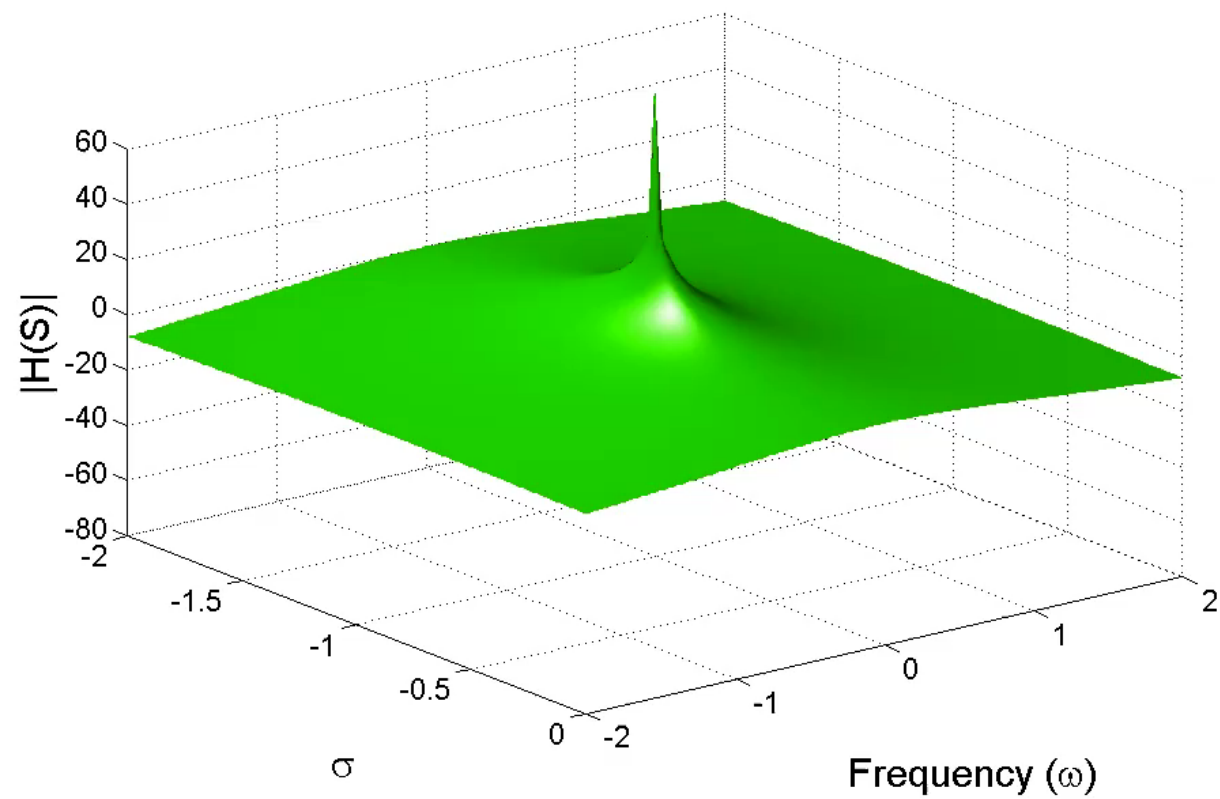
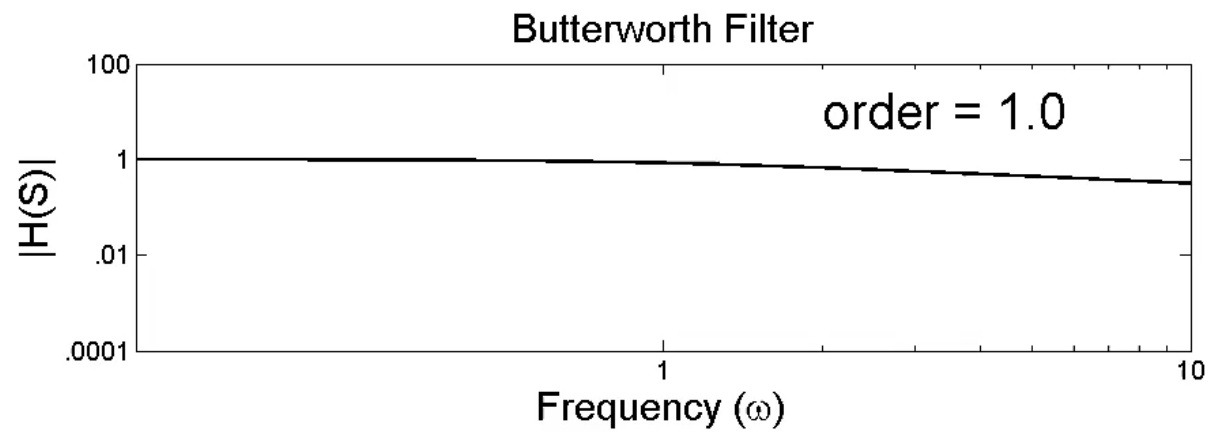
$K = 1, 2, \dots N$ = pole pair number

N = number of poles

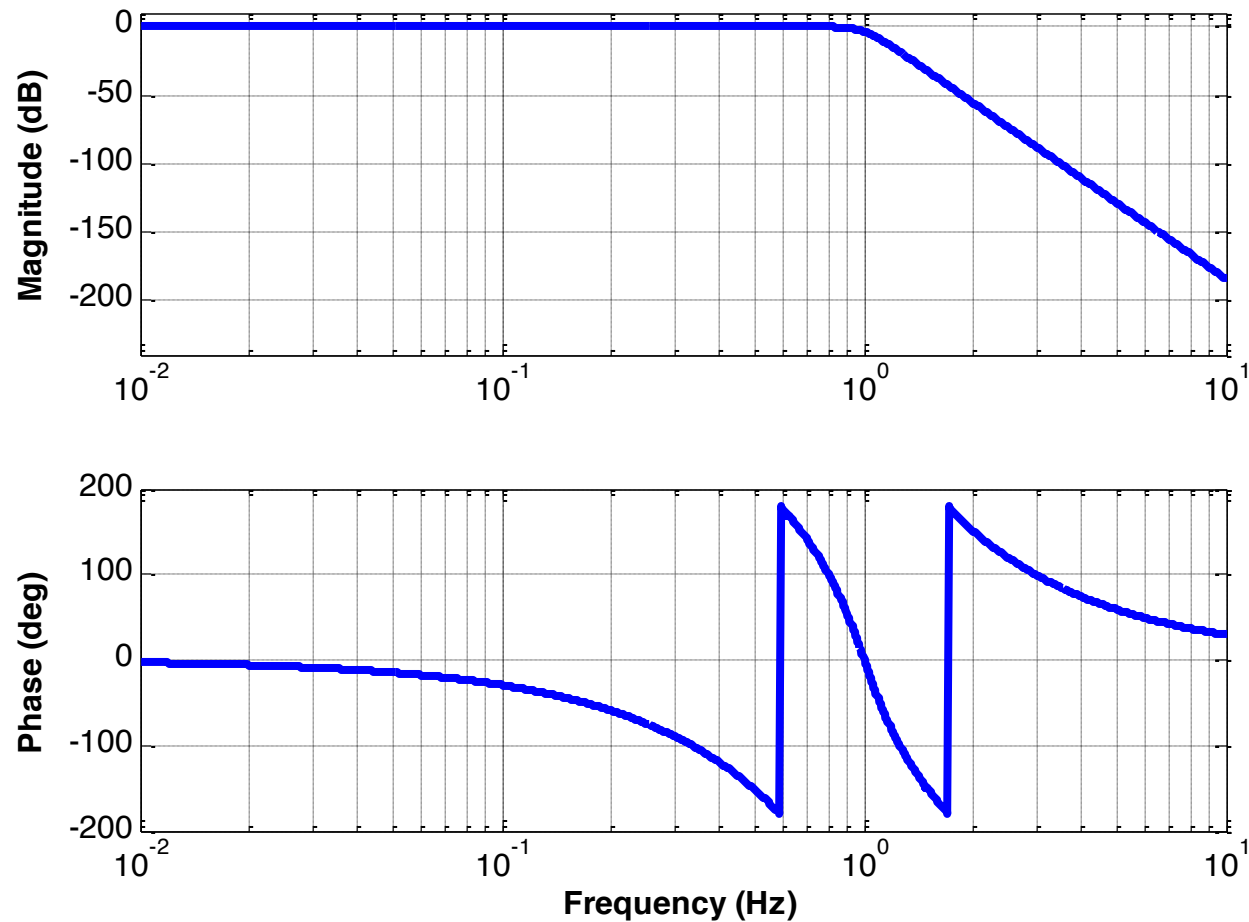


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8th order, Butterworth Low-Pass



Bessel Filters

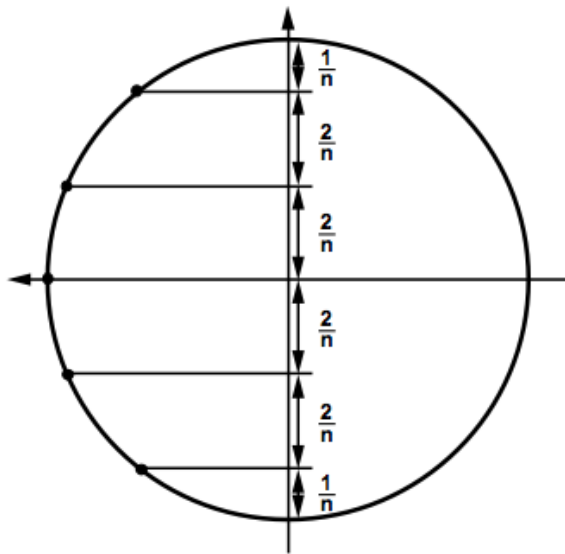
Poles arranged on a unit circle with
imaginary parts separated by:

$$\frac{2}{N}$$

where,

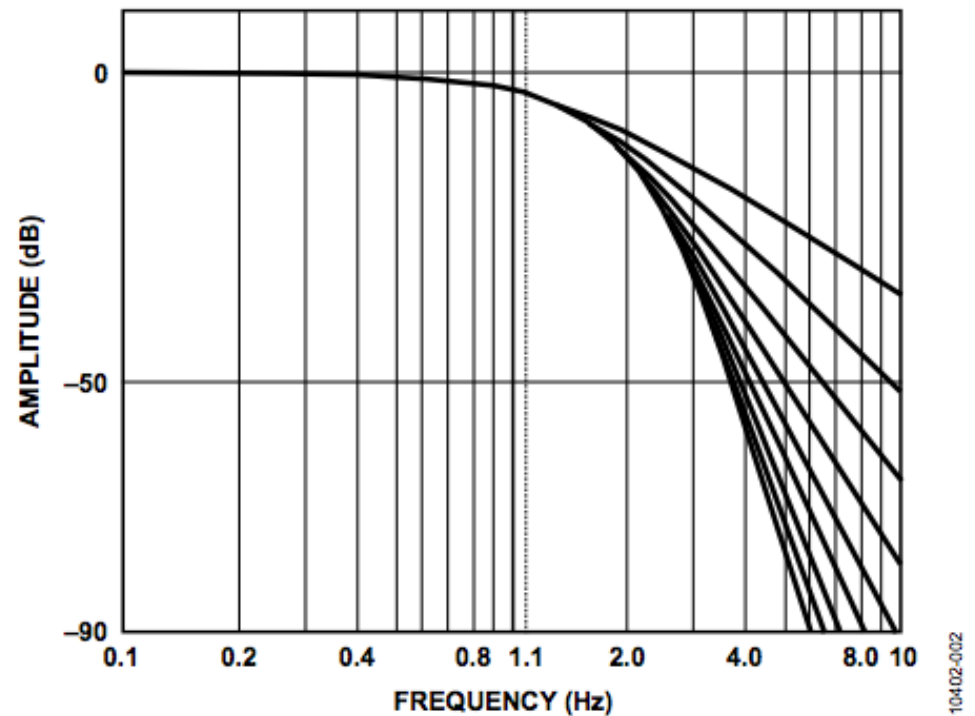
$K = 1, 2, \dots N$ = pole pair number

N = number of poles



10402-001

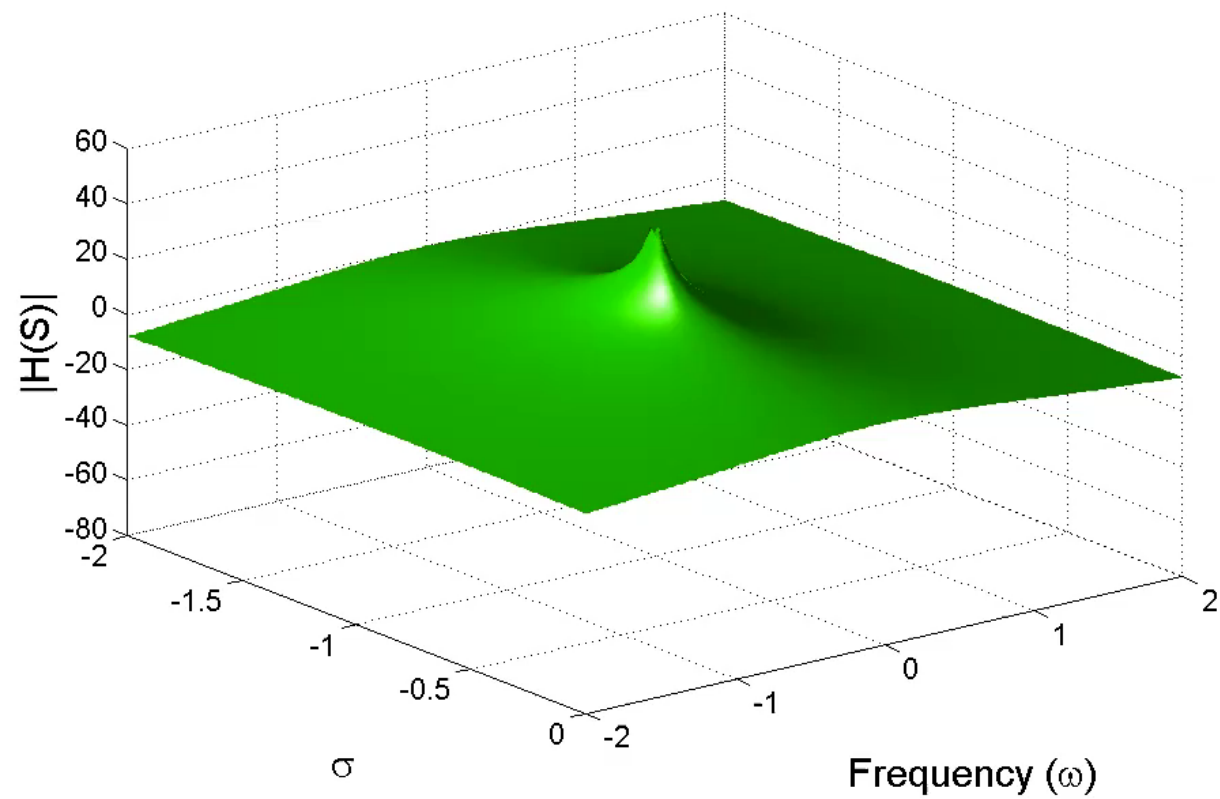
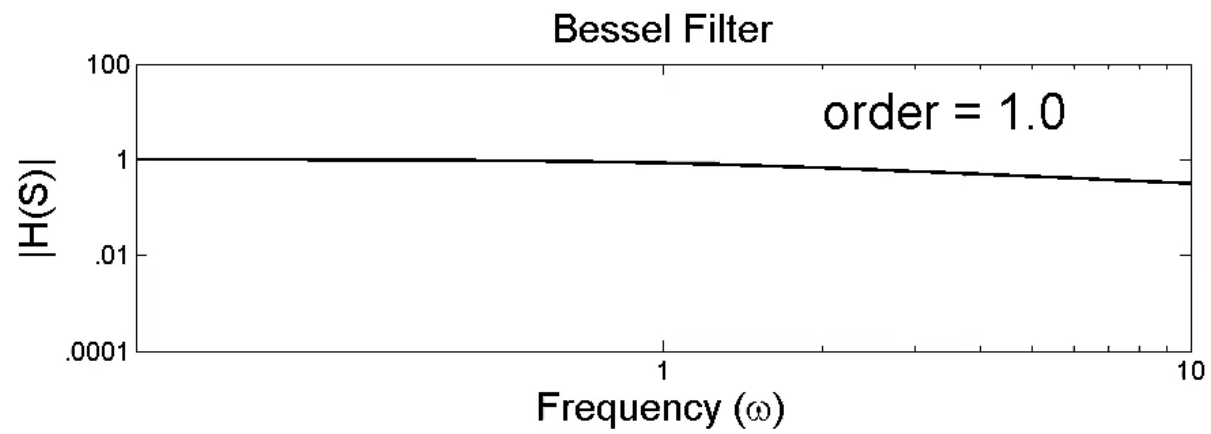
BESSEL RESPONSE



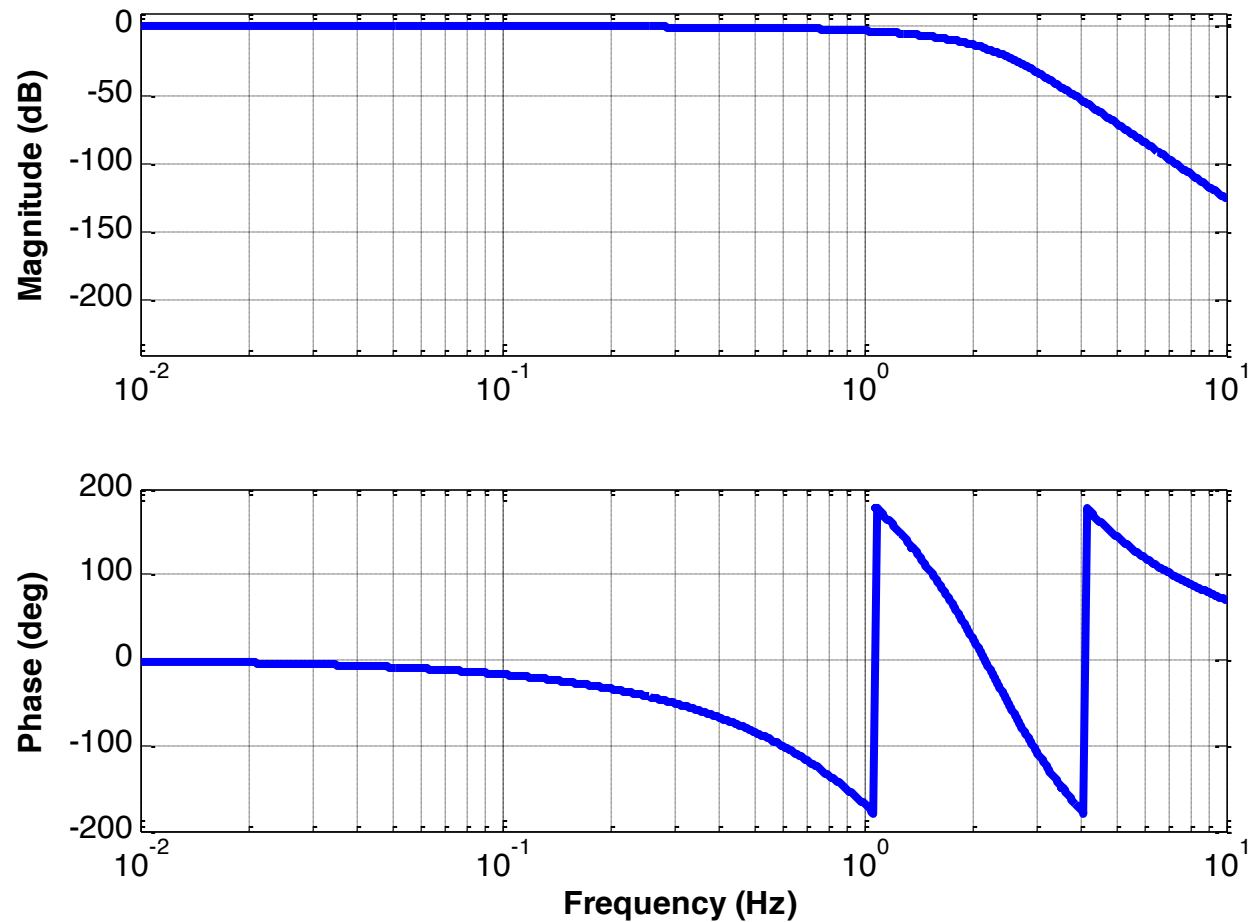
10402-002



Analog Devices MT-204



8th order, Bessel Low-Pass



Chebyshev Filters

Poles arranged on an ellipse at by multiplying

Butterworth (unit circle) pole positions:

$$-\sin \frac{(2K-1)\pi}{2N} \pm j \cos \frac{(2K-1)\pi}{2N}$$

by coefficients:

$$K_R = \sinh(A) \quad \text{and} \quad K_I = \cosh(A)$$

where,

$$A = \frac{1}{N} \sinh^{-1} \left(\frac{1}{\varepsilon} \right)$$

as,

$$-K_R \sin \frac{(2K-1)\pi}{2N} \pm j K_I \cos \frac{(2K-1)\pi}{2N}$$

where,

N = number of poles

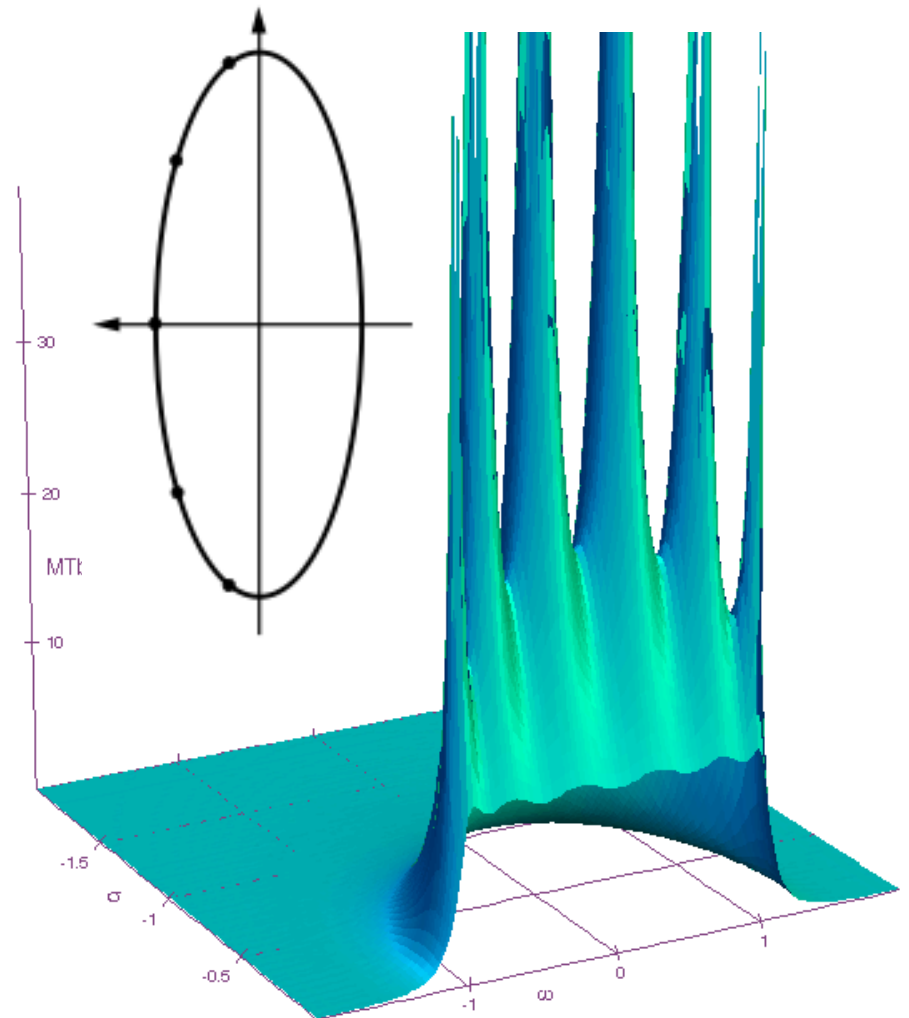
and,

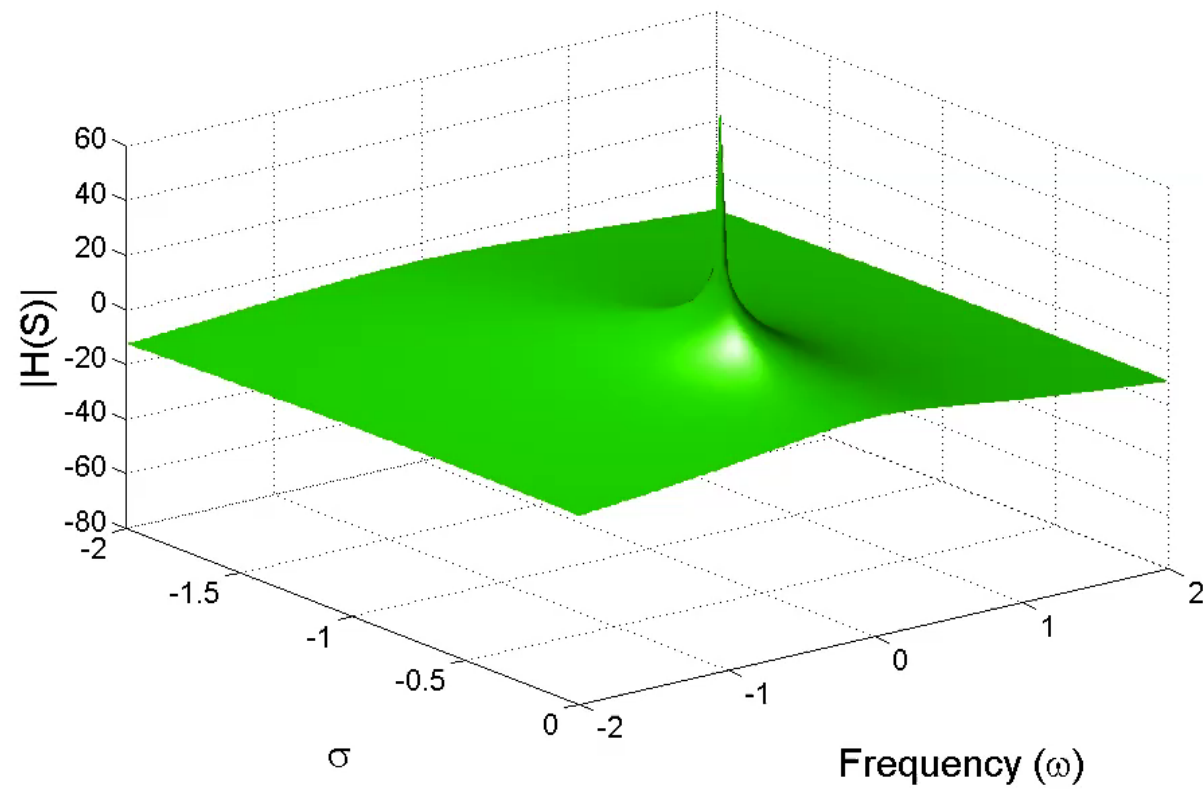
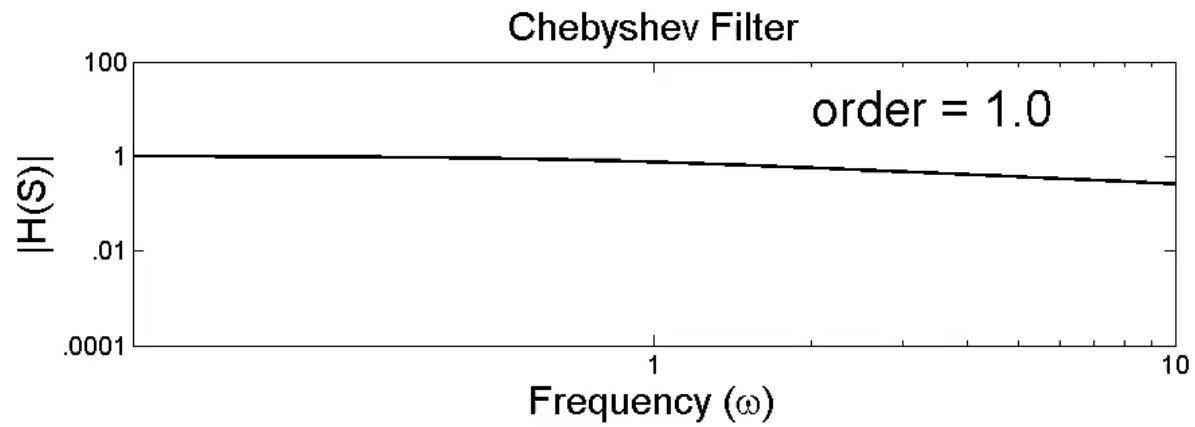
$$\varepsilon = \sqrt{10^R - 1} \quad \text{with} \quad R = \frac{R_{dB}}{10}$$

where,

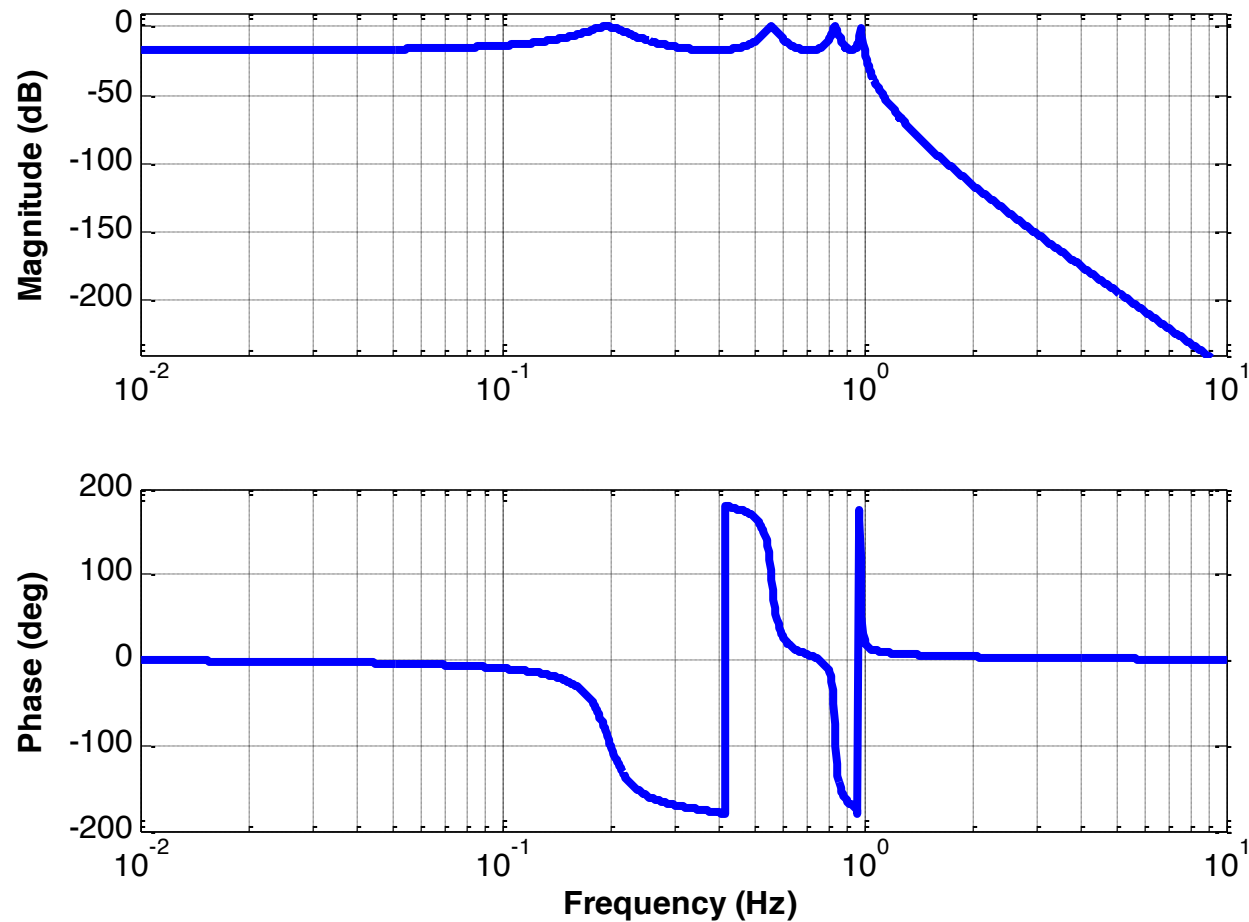
R_{dB} = pass-band ripple in dB

K = 1, 2, ... N = pole pair number

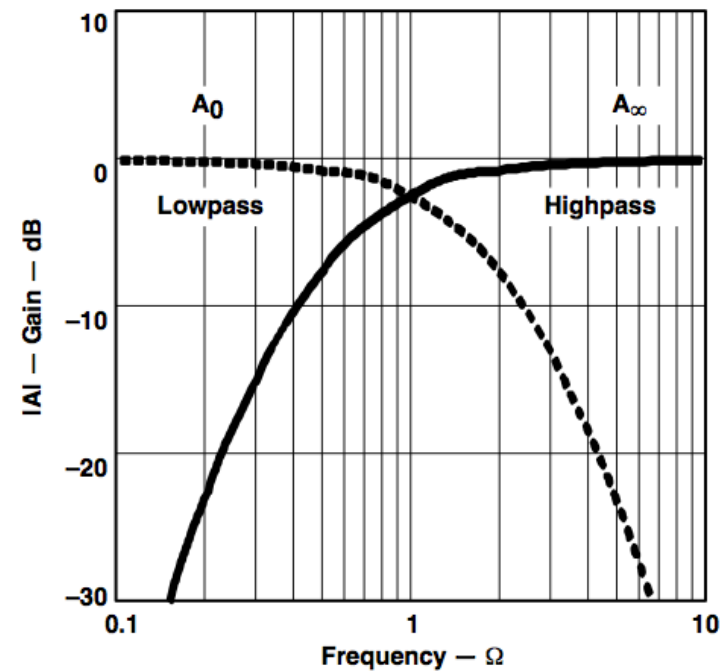
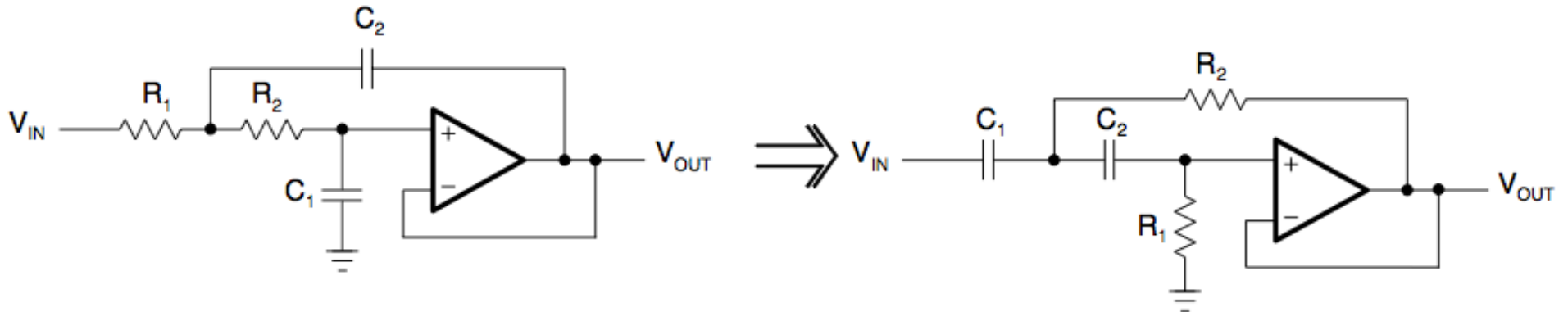




8th order , Chebyshev Type I Low-Pass



Low- to High-Pass Via Component Exchange



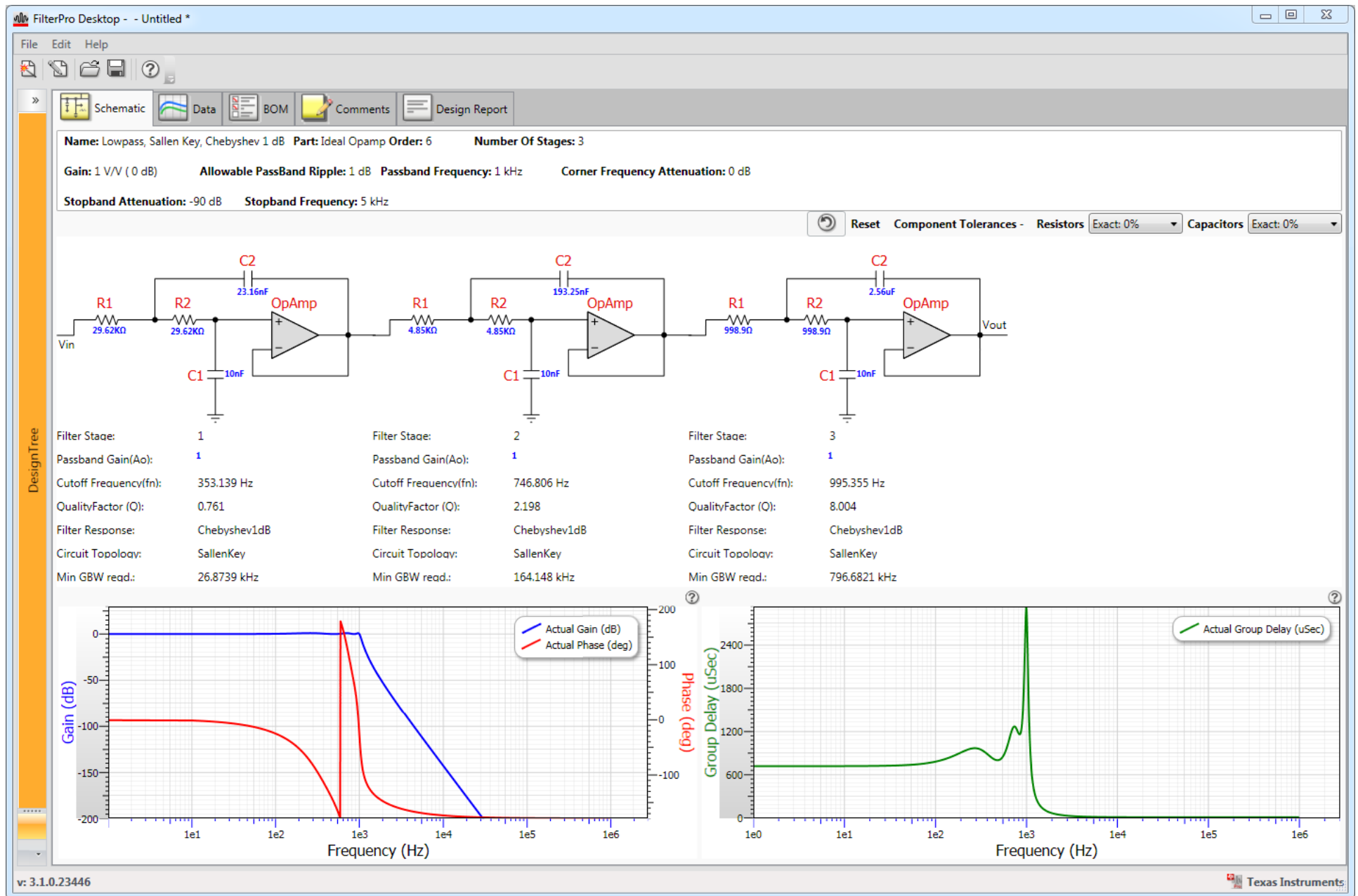
<http://www.ti.com/lit/ml/sloa088/sloa088.pdf>



Filter Design Tools

- Excellent application (Windows only, runs on Parallels) : <http://www.ti.com/tool/filterpro>
- www.analog.com/Analog_Root/static/techSupport/designTools/interactiveTools/filter/filter.html
- Useful Applet:
<http://www.analog.com/designtools/en/filterwizard/#/type>
- A complete listing of very many tools (user be ware!) can be found at Circuit Sage:
<http://www.circuitsage.com/filter.html>





Switched-Capacitor Filters

- **Basic idea**
- **Example and a bit of math**
- **Linear chips 8th order and tuneable...**



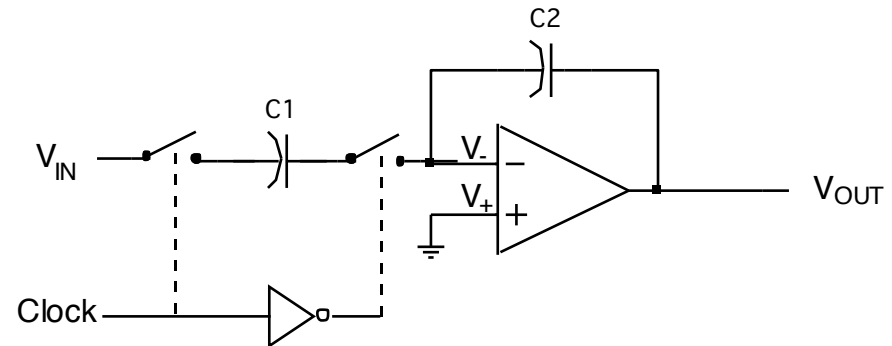
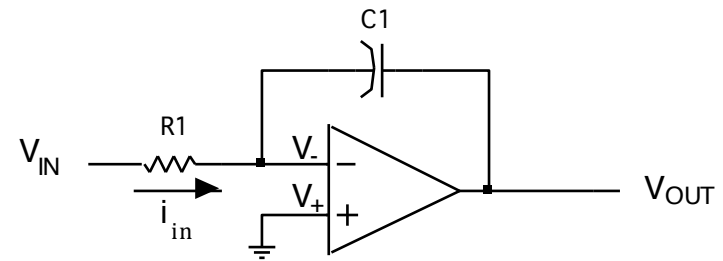
Switched Capacitor Filters

- It is possible to build analog filters where instead of resistors to make RC time constants, analog switches and capacitors are used to “simulate” resistances.

$$R = \frac{V}{I} = \frac{1}{Cf}$$

- In some filter types (state-variable or biquad), built with integrators, the integrator gain controls the cutoff frequency - therefore, *you can sweep the cutoff frequency with the clock frequency.*

$$V_{out} = -\frac{1}{R_1 C_1} \int V_{in} dt$$



$$V_{out} = -f_{clock} \frac{C_1}{C_2} \int V_{in} dt$$

http://en.wikipedia.org/wiki/Switched_capacitor



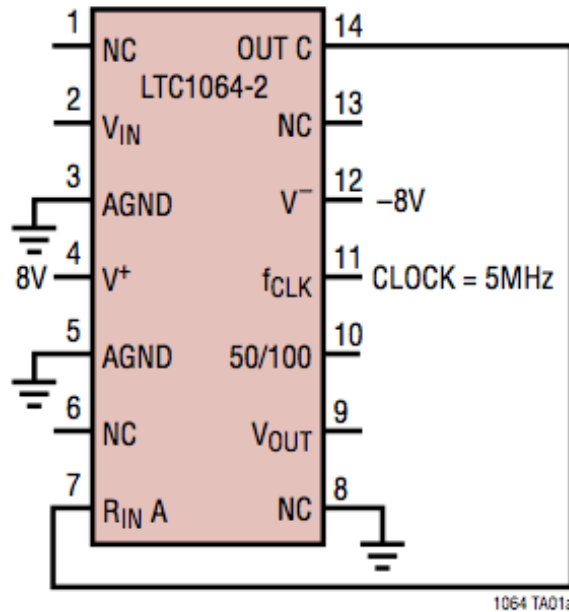
Switched Capacitor Filters

- Complex filters thus become tunable, but at the cost of power consumption.
- *Consider how hard it would be to tune an 8th-order Butterworth filters built with op-amps, resistors and capacitors!*
- Another issue with S/C filters is feedthrough of some of the clock signal, which often requires a simple “post-filter” to remove these components, typically much higher in frequency than the frequencies passed by the filter.
- This can be an issue for high-pass filters implemented with S/C methods though.



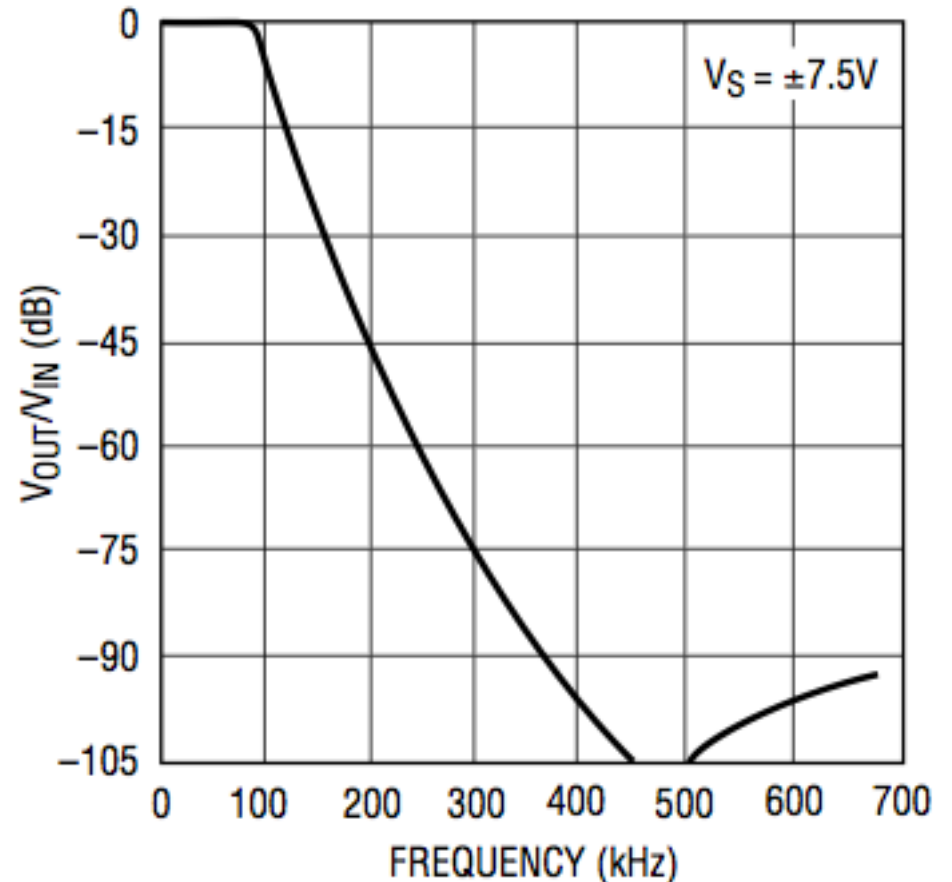
LTC1064 Switched Capacitor Filter

**8th Order Clock Sweepable
Lowpass Butterworth Filter**



NOTE: THE POWER SUPPLIES SHOULD BE BYPASSED BY A 0.1 CAPACITOR CLOSE TO THE PACKAGE. THE NC PINS 1, 6, 8, AND 13 SHOULD BE PREFERABLY GROUNDED.

Measured Frequency Response



Source: Linear Technology LTC1064 Datasheet.

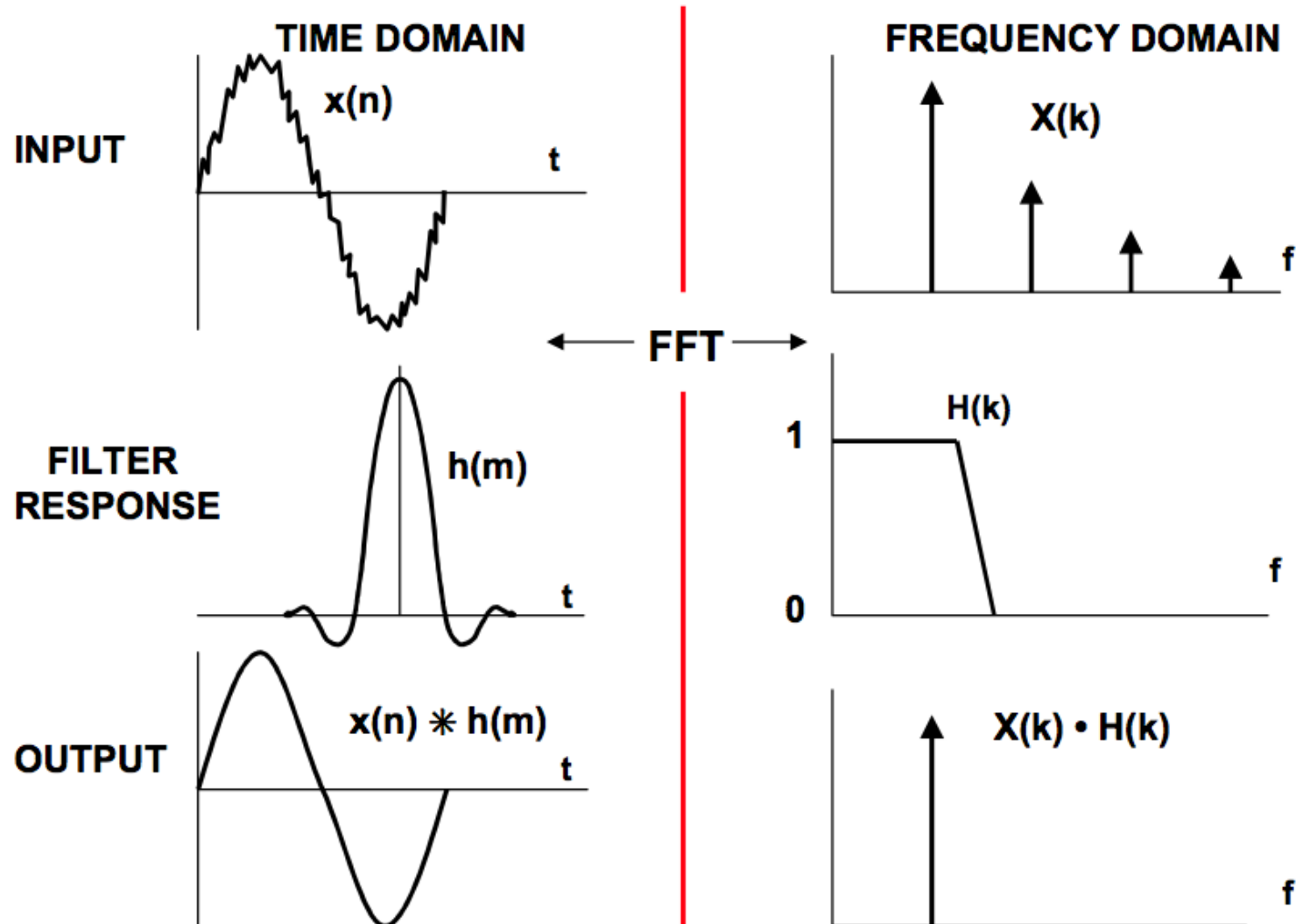


Digital Filters

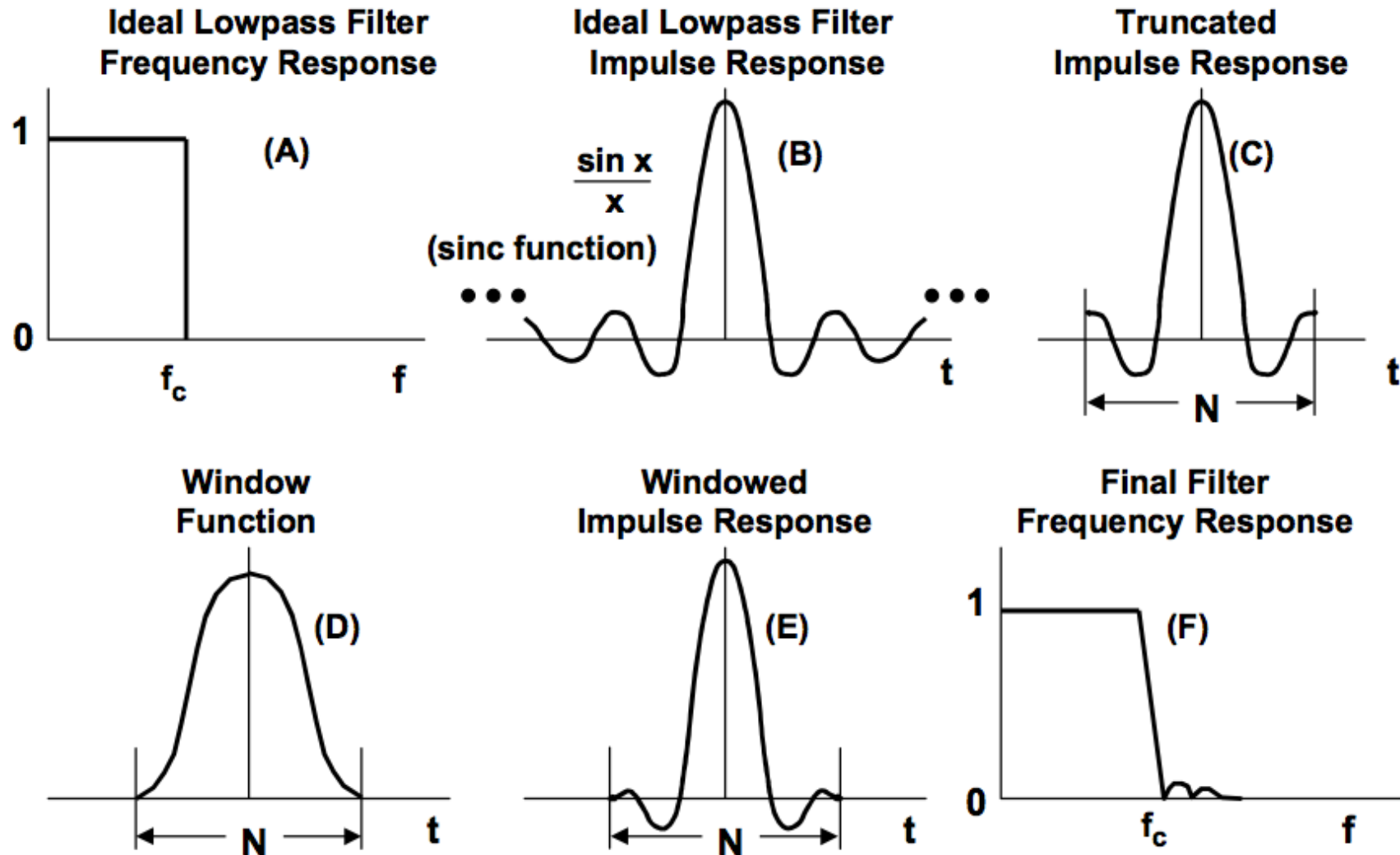
- Filters can be implemented in the digital domain that rival or exceed the performance of physical counterparts.
- The basic principle is to transform the transfer function $H(S)$ into a digital form based on discrete time delay units, not continuous time: $H(z)$.
- One way of thinking of this is to imagine implementing a filter by convolving a digital version of its impulse response with an incoming stream of digitized analog data – the resulting output data would be filtered.



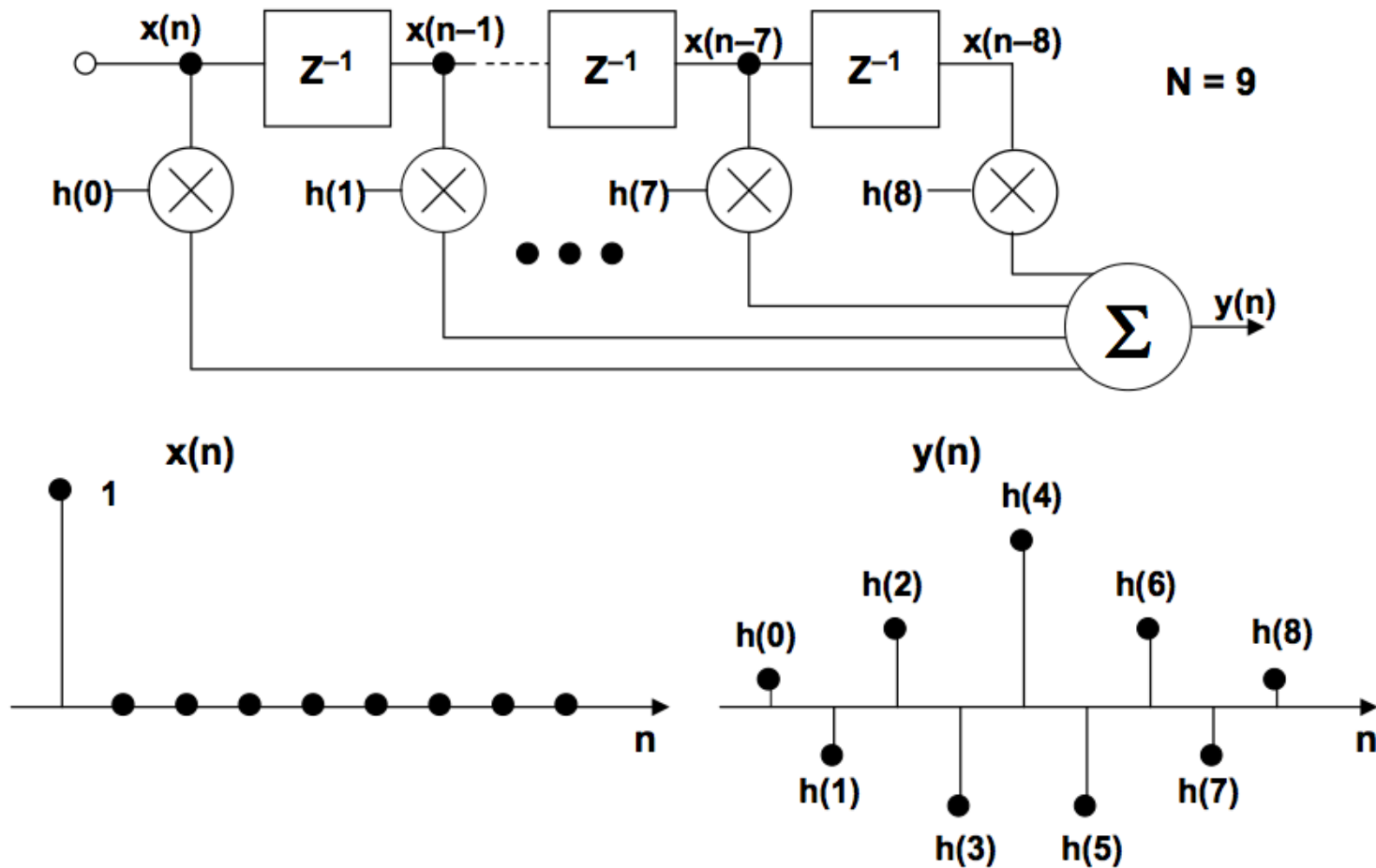
Duality of Time and Frequency



Windowed-SINC Design Method



Implementation (FIR Filter)



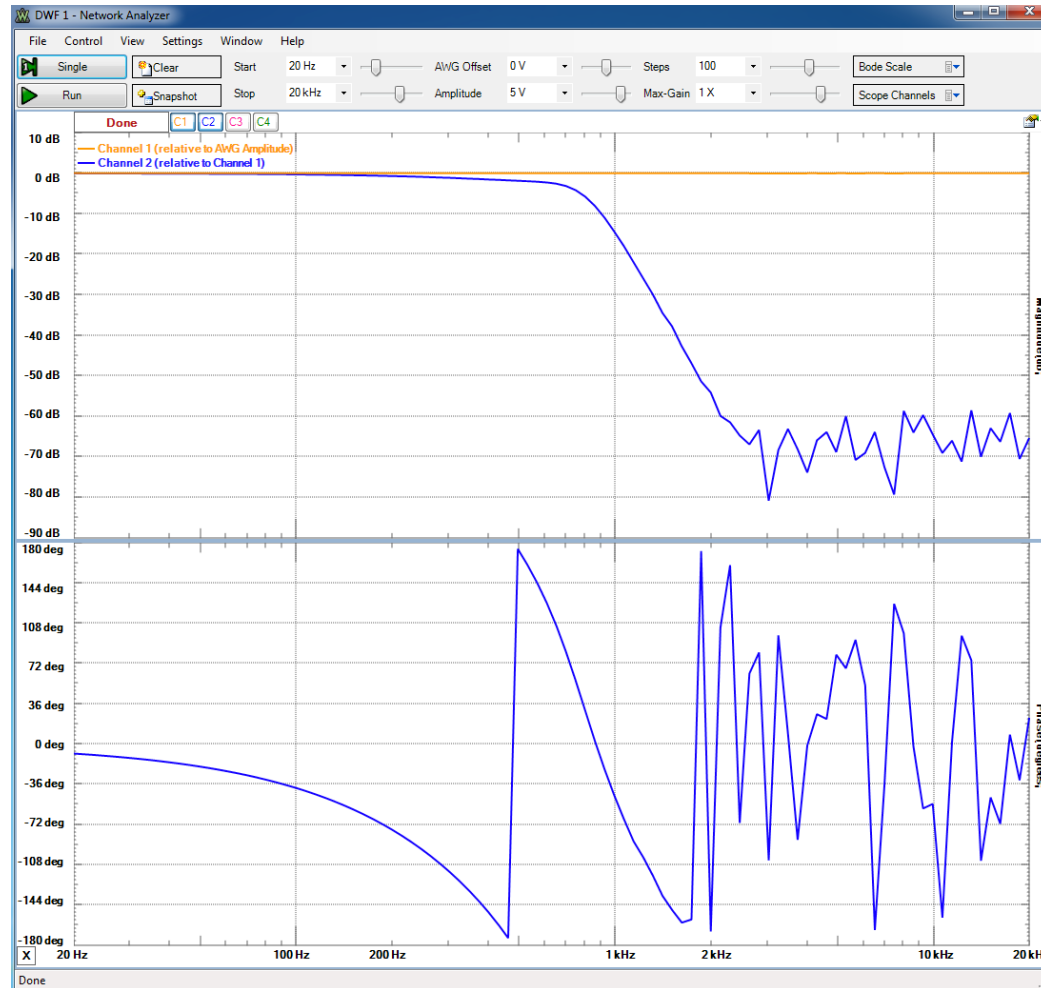
Types of Digital Filters

- **Finite Impulse Response (FIR), Infinite Impulse Response (IIR), adaptive, etc.**
- **Easy to re-program, highly versatile.**
- **Often requires more power than analog or switched capacitor implementations.**
- **Details in other courses, but for now just need to understand the basics for the upcoming lab.**

http://en.wikipedia.org/wiki/Digital_filter
<https://ccrma.stanford.edu/~jos/filters/>



Issues With Measurements



Digilent Analog Discovery and Electronics Explorer systems currently have dynamic range limitations in “Network Analyzer” mode that show up as “noise” when the signal level coming from the filter becomes too low.



SUMMARY

- **Hopefully now you have a sense of the different ways a filter can be implemented (e.g., LC, active, switched capacitor and digital).**
- **The real-world implementation, meaning the choice of which type of filter, can involve trade-off of size, weight, power and cost, as well as tunability, field-reprogrammability, precision, manufacturability, amplitude range, etc.**
- **As in any type of engineering, navigating such trade-spaces is critical to good design.**



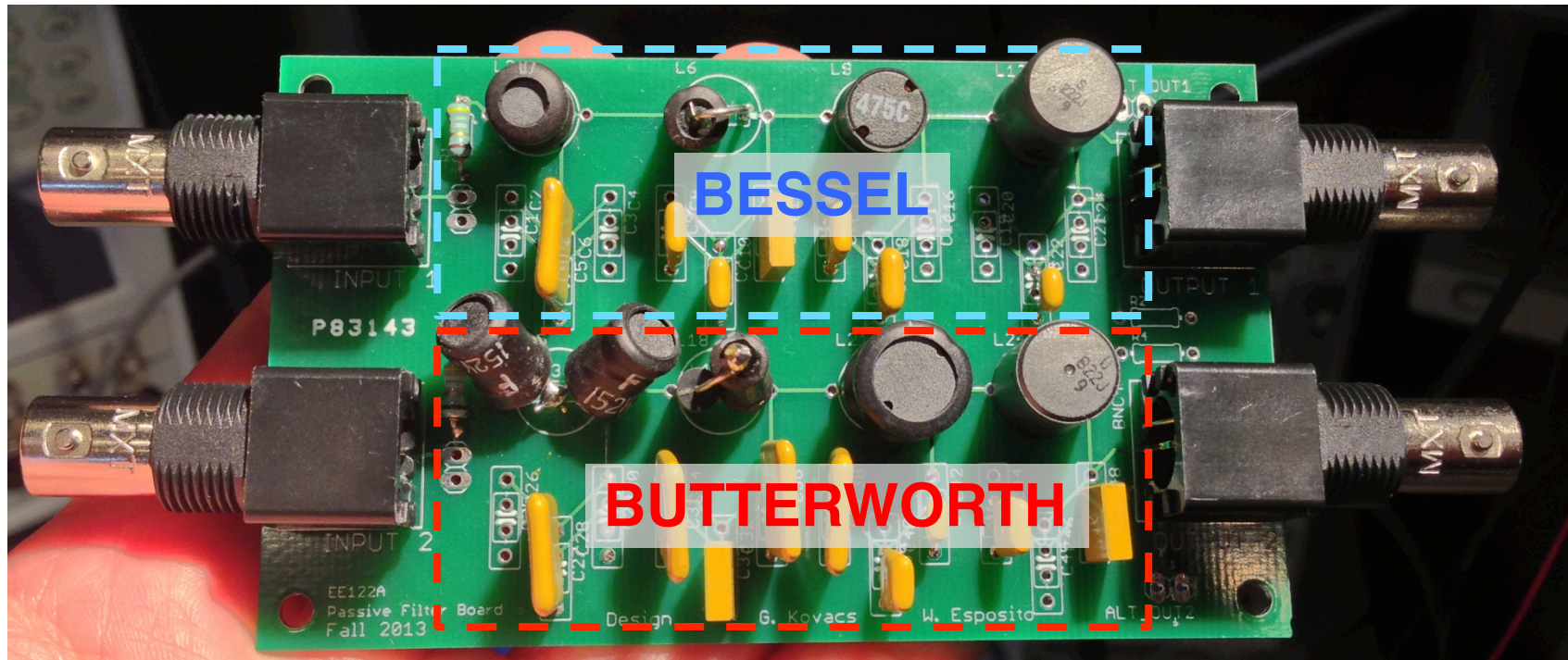
Appendix 0: Notes for Lab 3

**Some helpful information to supplement
the prelab/lab handout.**





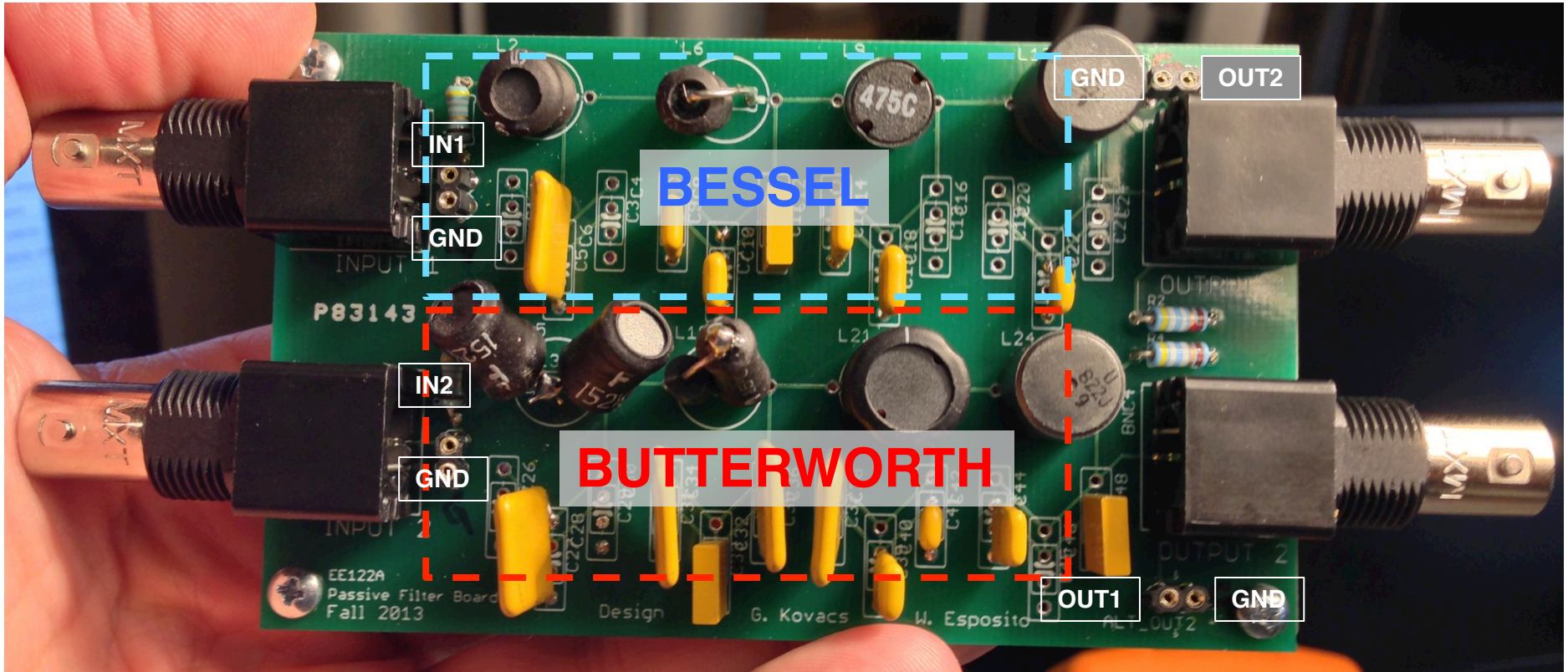
LC Filter Board



- Two channels, different filter topologies.
- Channel 1: Bessel, 8th Order, 1 kHz LPF.
- Channel 2: Butterworth, 8th Order, 1 kHz LPF.
- 50 Ω input and output and output terminations on board.



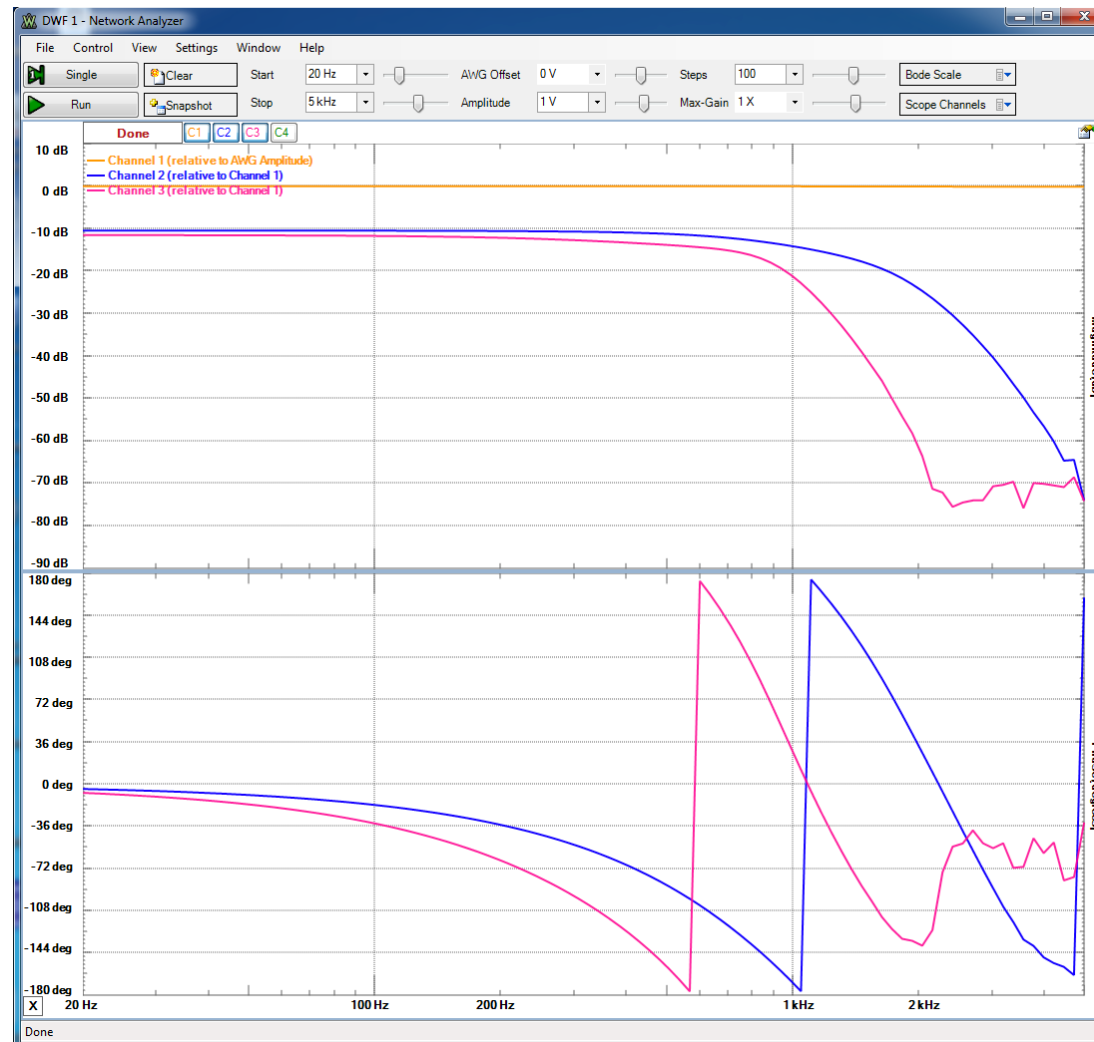
LC Filter Board



Note input and output connectors (female pin sockets) for signal and ground locations (not obvious).



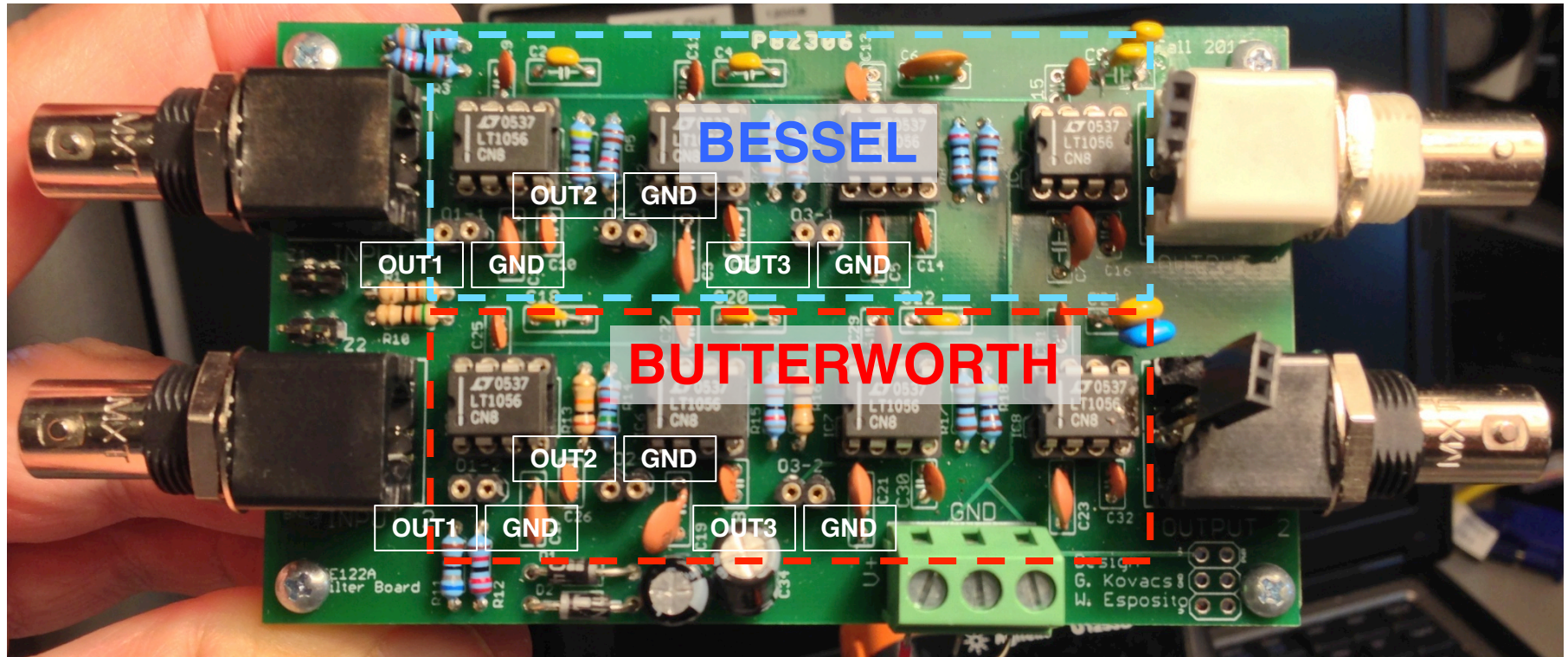
LC Filter Board Frequency Response



Red trace = Butterworth, Blue trace = Bessel. Both were designed for 1 kHz cutoff and are built with approximate component values.



Op-Amp Filter Board



Note which filter is which, and where the clock adjust potentiometer is located (you probably do not need to touch it). Note marked outputs for the first three of four op-amp stages (the fourth is the final output at right) so you can investigate the intermediate signals.

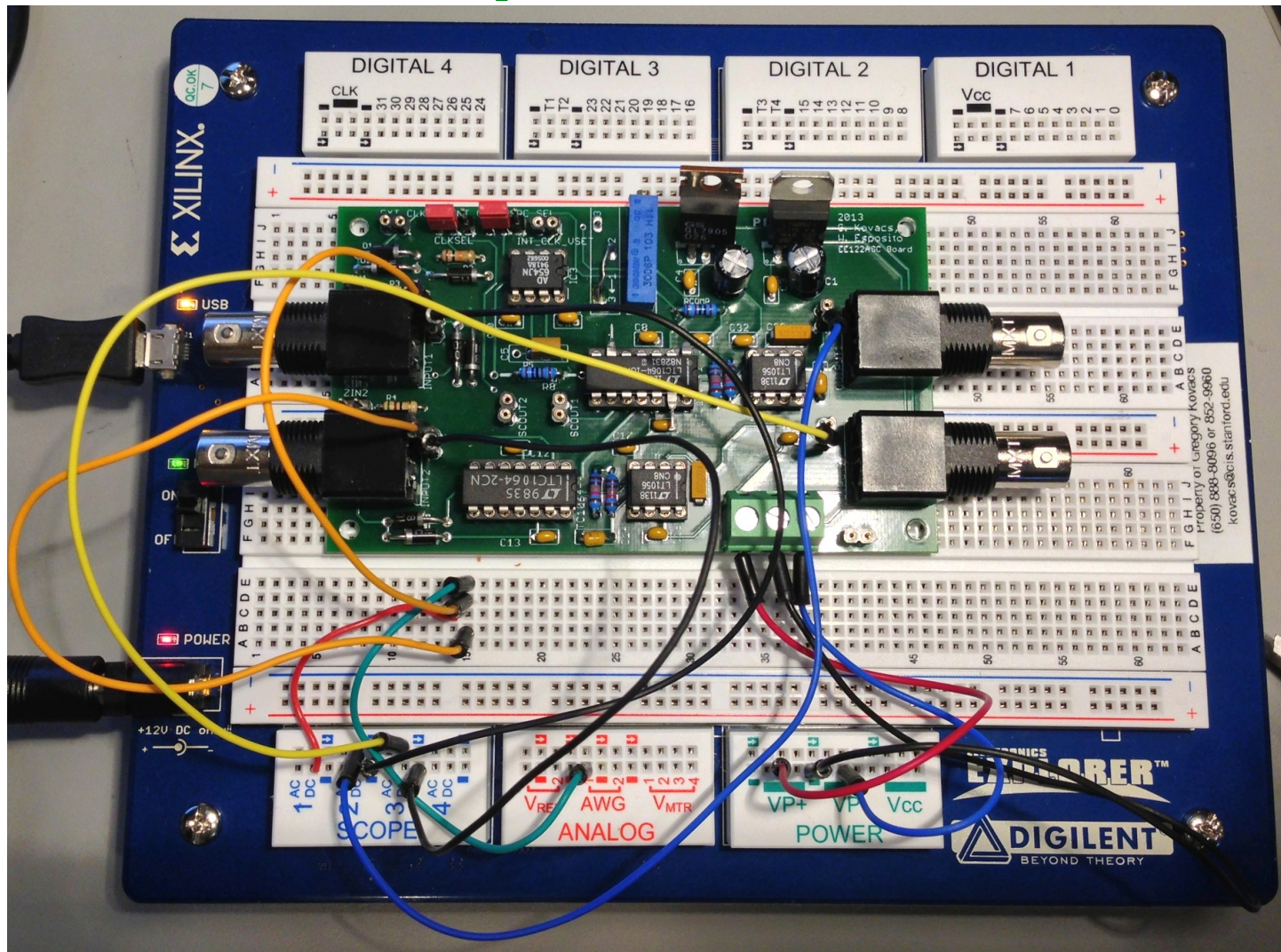


Switched Capacitor Filter Board

- Two channels, different filter chips.
- Channel 1: LTC1064-1 8th Order Elliptic LPF
- Channel 2: LTC1064-2 8th Order Butterworth LPF
- Needs $\pm 9\text{V}$ input power at screw terminal block.
- Clock for filters adjusted by turning potentiometer (R6) and should be set for 100 kHz, which gives cutoffs for both filters of 1 kHz.
- Input impedance jumpers (near input BNC's) should be removed for Electronics Explorer board or put in place if 50 Ω termination is desired.
- Note that the female pin headers (places to plug in wires) at the inputs and outputs have grounds at the bottom and top of the pairs of holes, respectively.



Switched Capacitor Filter Board

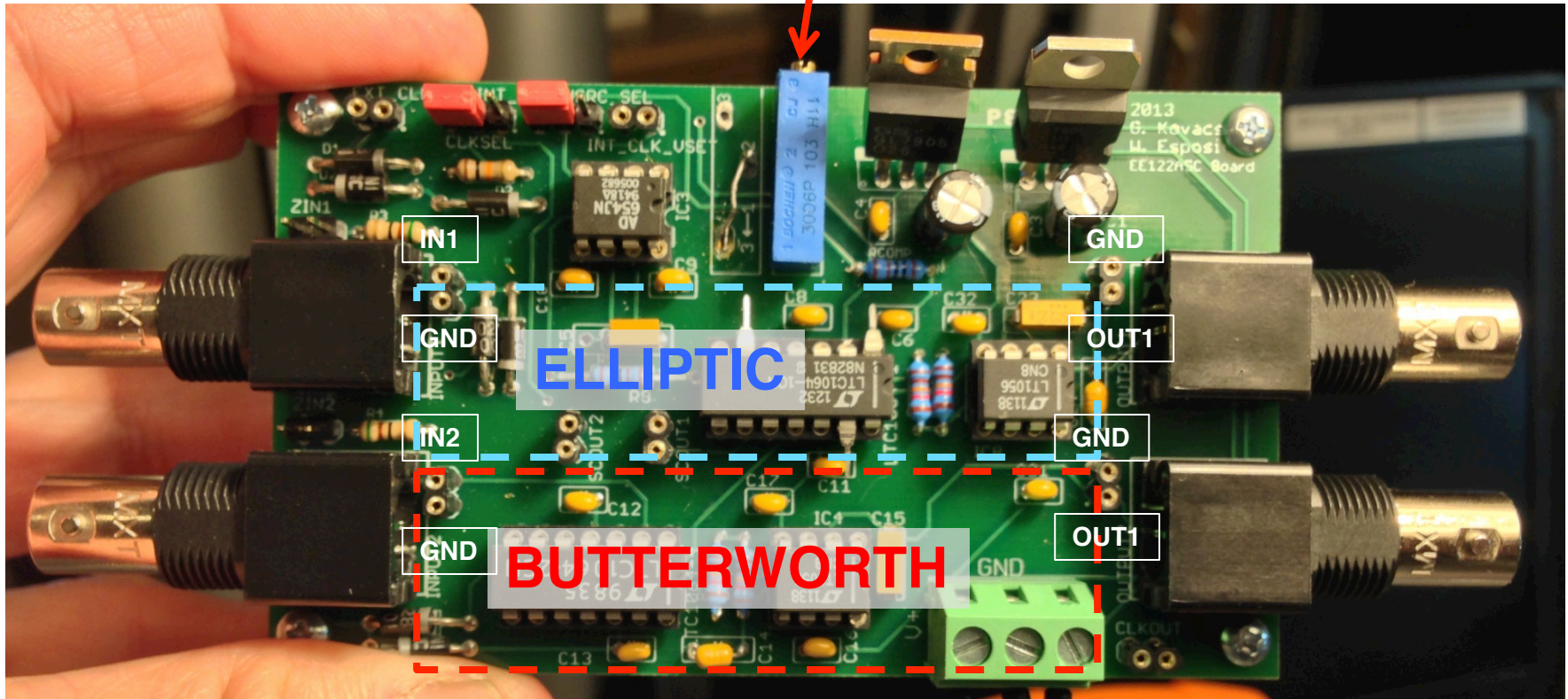


- Note connections in the example: as used with Electronics Explorer. Only one ground wire is required, but extras were added here.



Switched Capacitor Filter Board

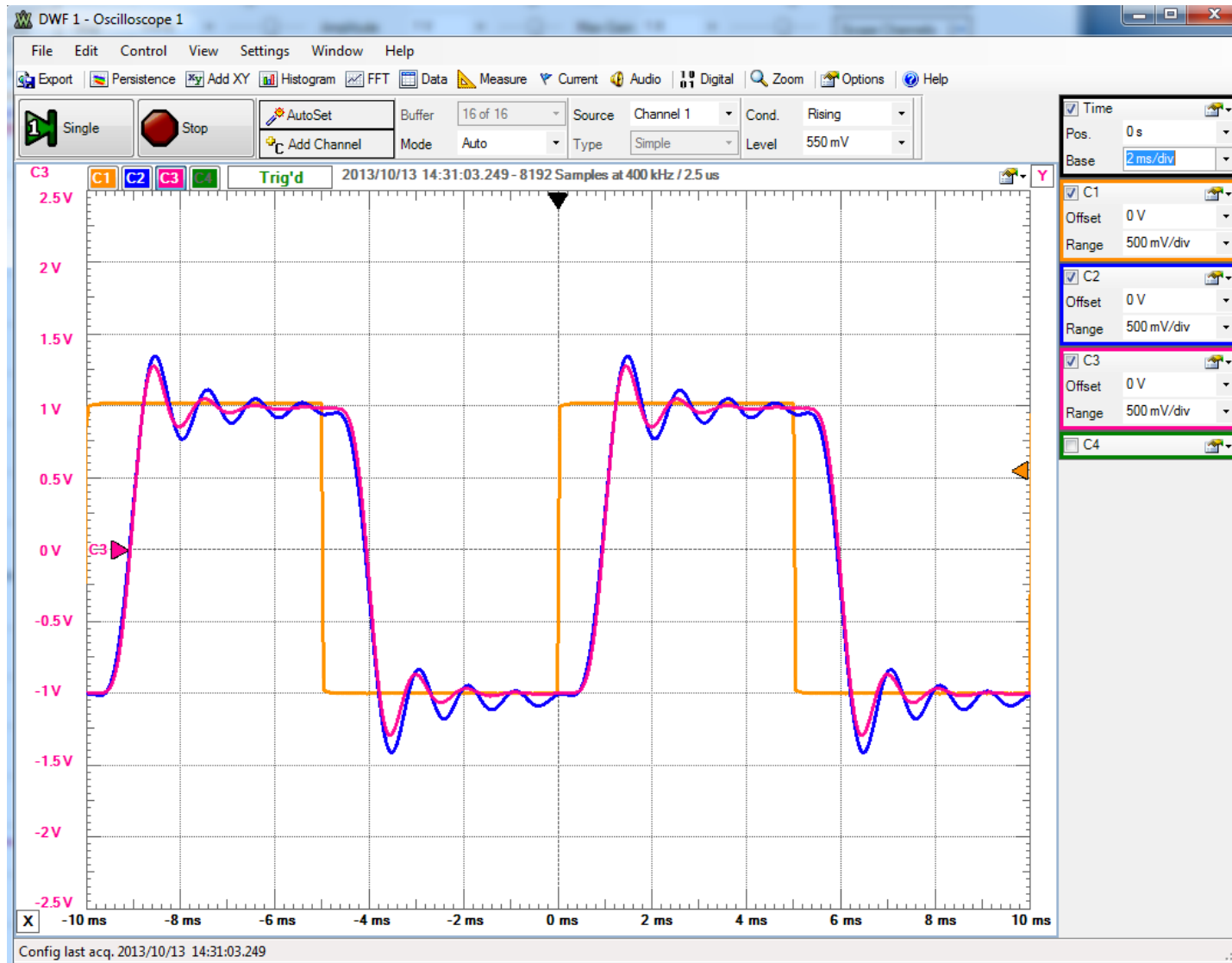
Clock Adjust Pot



Note which filter is which, and where the clock adjust potentiometer is located (you probably do not need to touch it).



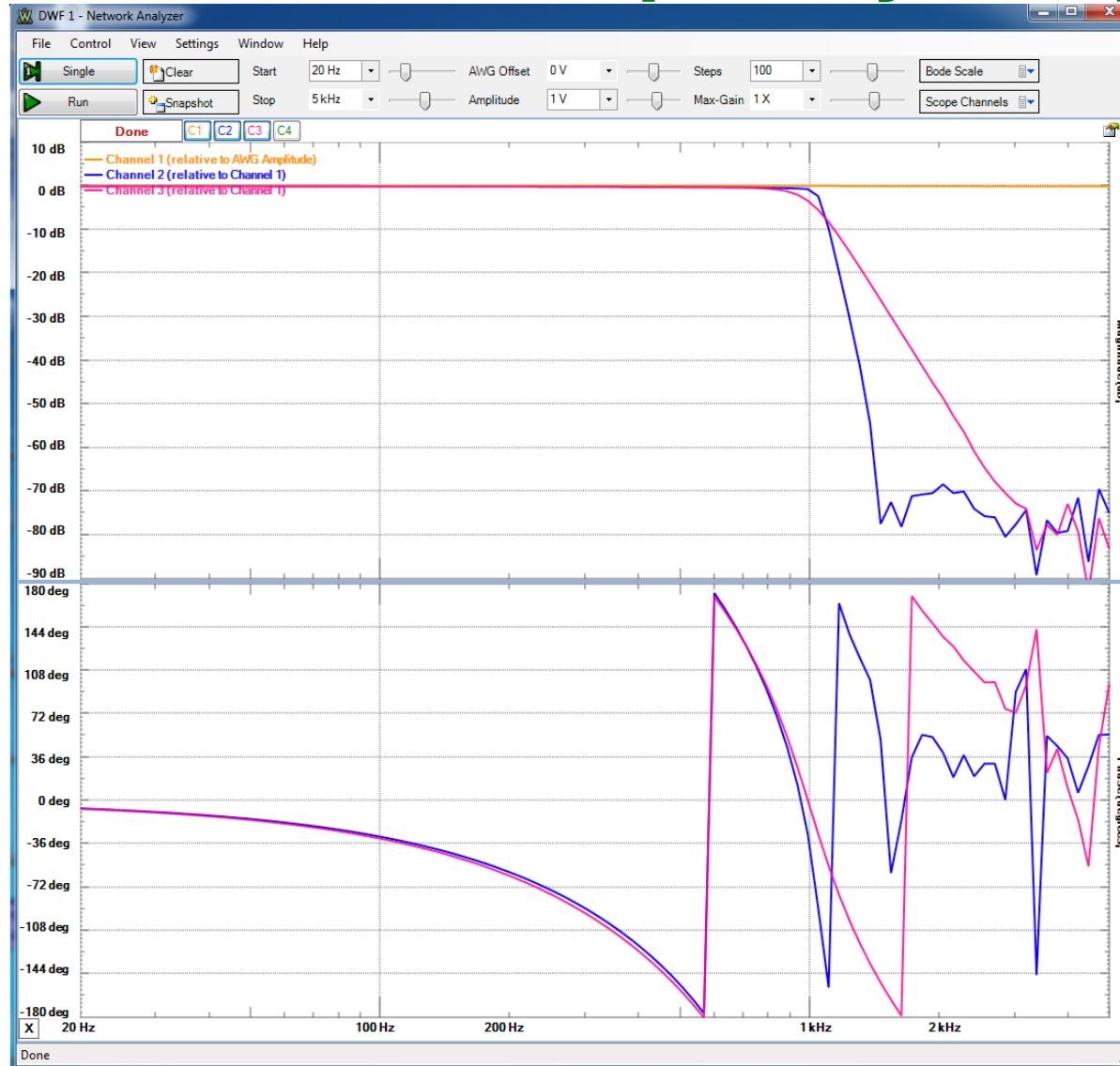
S/C Filter Board Step Response



Response to 100 Hz squarewave. Red trace is Butterworth, blue trace is Elliptic response.



S/C Filter Board Frequency Response



Gain and phase response, 20 Hz – 5 kHz, 1V input amplitude, $\pm 9\text{V}$ supplies. Red trace is Butterworth, blue trace is Elliptic response.



Appendix 1: Scalable Filter Design

Not used much anymore, but can be handy in a pinch.



SCALABLE FILTER DESIGN

- NOT ALWAYS APPLICABLE, BUT CAN BE USED IN MANY CASES
- SIMPLE DESIGN PROCESS!
- STANDARD FILTER DESIGN IS NORMALIZED TO A PARTICULAR CUT-OFF FREQUENCY AND IMPEDANCE
- YOU SCALE THE FILTER COMPONENTS TO MEET YOUR REQUIREMENTS

THE BASIC IDEA:

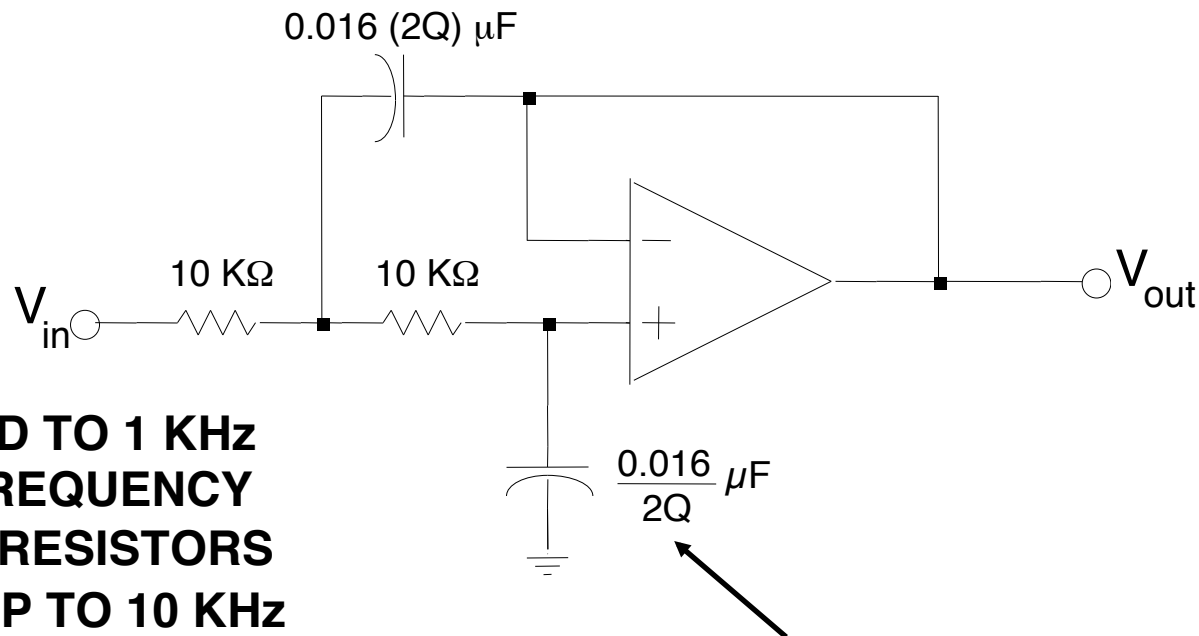
TO CHANGE CUT-OFF FREQUENCY, SCALE RC TIME-CONSTANT

TO CHANGE IMPEDANCE SCALE R's and C's IN INVERSE PROPORTION
TO KEEP RC TIME-CONSTANT THE SAME

TO HUNT FOR "REAL" COMPONENT VALUES, SCALE RC TIME-
CONSTANT AND THEN SCALE IMPEDANCE



SCALABLE LOW-PASS FILTER



- **NORMALIZED TO 1 KHz CUT-OFF FREQUENCY AND 10 KΩ RESISTORS**
- **TO SCALE UP TO 10 KHz CUT-OFF, YOU WANT RC TO GO DOWN BY 10....**
THEREFORE, EITHER REDUCE ALL C'S BY 10X OR REDUCE ALL R'S BY 10X
- **THEN ADJUST C'S TO GET DESIRED Q...**

CAPACITOR VALUES WRITTEN IN TERMS OF "2Q" TO EMPHASIZE RELATIONSHIP TO POLE POSITIONS!

$$\text{Pole Distance from } j\omega \text{ Axis} = \frac{\omega_o}{2Q} = \frac{2\pi f_o}{2Q}$$



EXAMPLE: SCALED LOW-PASS

- "NORMALIZED" DESIGN IS 1 KHz CUTOFF, 10 K Ω RESISTANCE
- WANT 300 Hz CUTOFF, 1 K Ω RESISTANCE

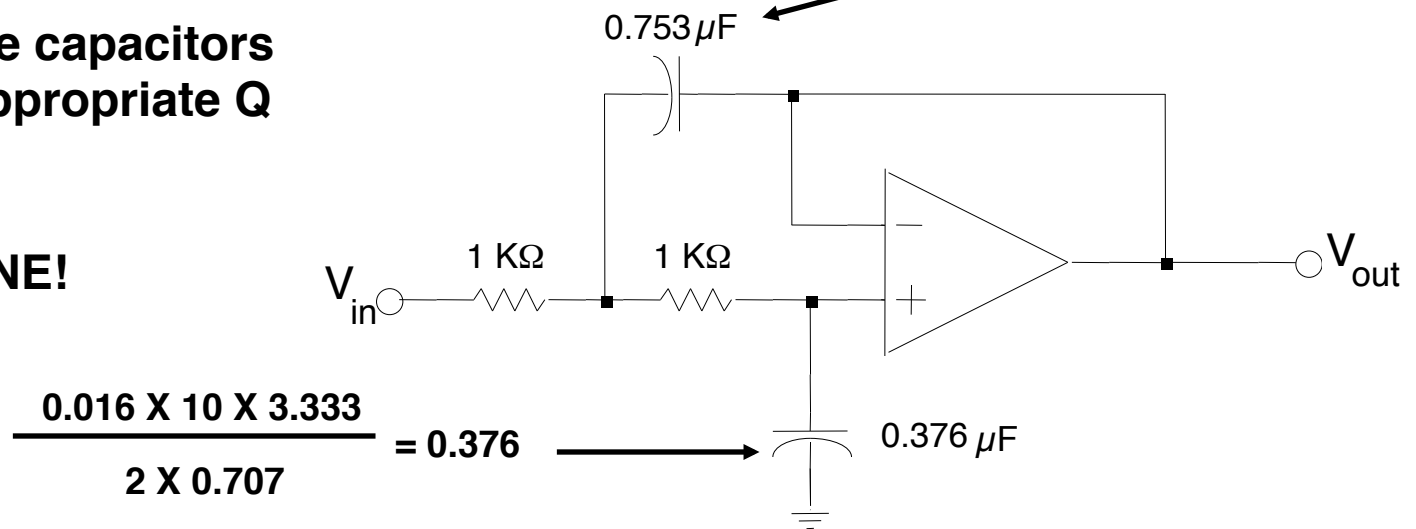
- 1) Replace 10 K Ω resistors with 1K Ω and multiply capacitances by 10 to keep RC time-constant the same (1 KHz cutoff).
- 2) Scale the cutoff frequency DOWN by increasing capacitors by:

$$RC_{\text{new}} = RC_{\text{old}} \left(\frac{1000 \text{ Hz}}{300 \text{ Hz}} \right)$$

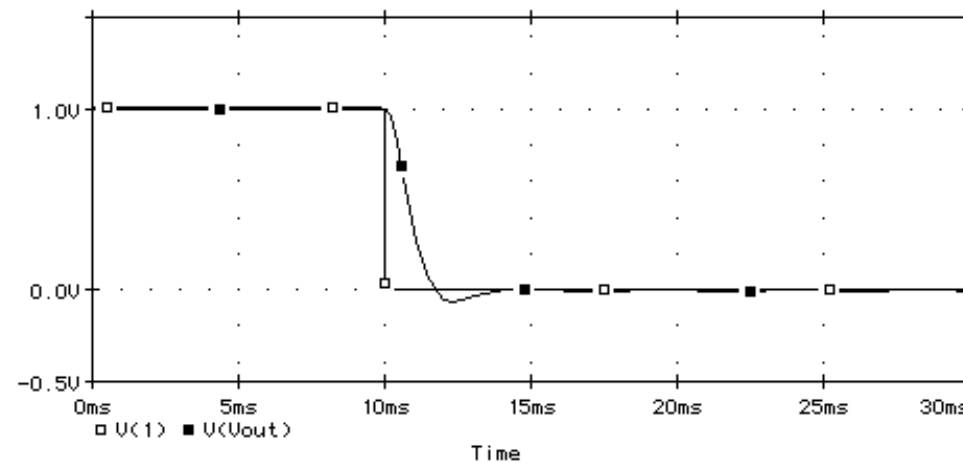
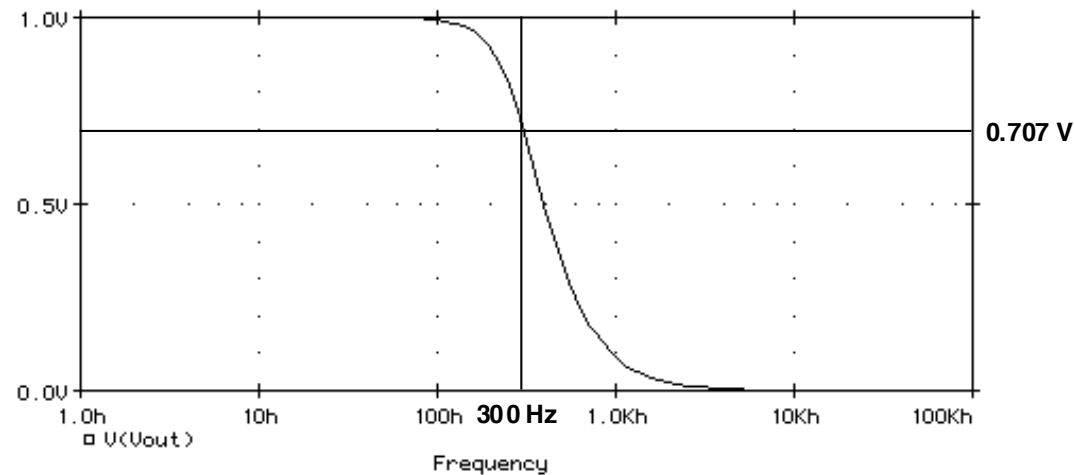
$$0.016 \times 10 \times 3.333 \times 2 \times 0.707 = 0.753$$

- 3) Scale the capacitors by the appropriate Q value....

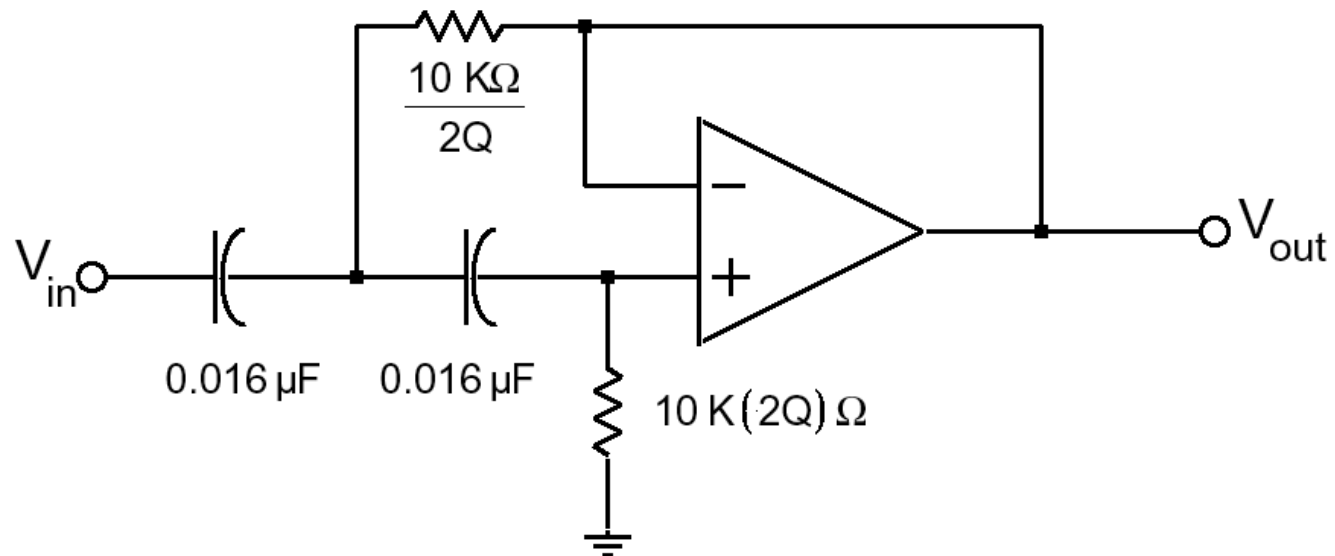
- 4) ALL DONE!



TESTING 300 Hz LPF WITH SPICE



SCALABLE HIGH-PASS



THE SAME TYPE OF SCALING, BUT A DIFFERENT FREQUENCY RESPONSE...

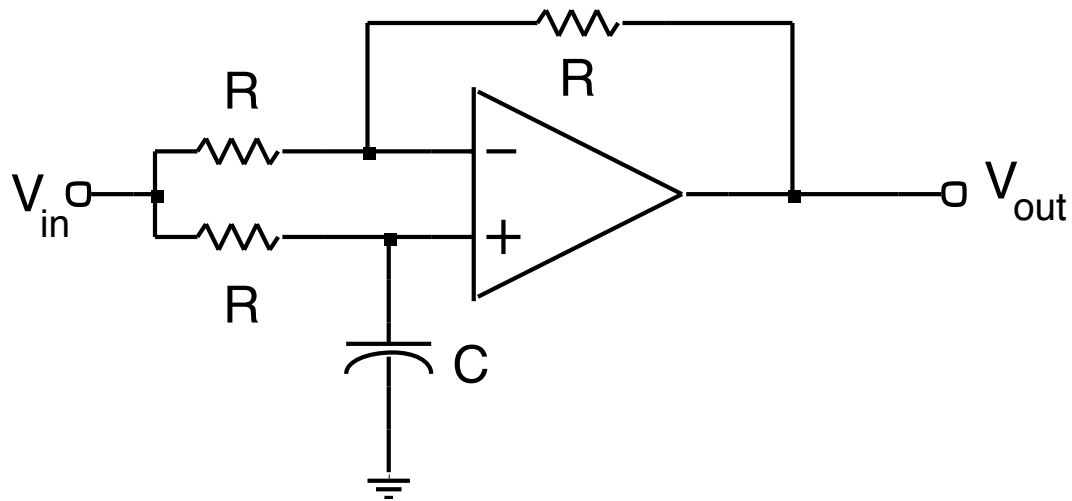
JUST SWAP THE R'S AND C'S (CAREFULLY)!



Appendix 2: All-Pass Filter



ALL-PASS? MAYBE IN LIBERAL ARTS. THIS IS EE!



$$H(s) = \frac{1 - RCS}{1 + RCS}$$

- What good is it? Can be used to match phase between audio channels.
- Passes all frequencies equally.
- Phase response varies with frequency.



Appendix 3: The Geophone



THE GEOPHONE

- **Geophones are used to detect very small vibrations of the earth, and can also detect fairly large vibrations (such as earthquakes).**
- **A geophone essentially a permanent magnet suspended by a spring within a cylindrical coil of wire.**
- **When the magnet is moved due to vibrations, its lines of magnetic flux cut through the wires - moving magnetic fields induce currents in wires.**
- **Geophones generally have second-order frequency responses, with 3 dB frequencies in the low Hz range.**



